

2000

II| : $4a^3/3$

2. (a) (i) 6, 1/5 (ii) infinity (iii) zero (b) (i) $2 \sin x$ (ii) $x=0: 1, x \neq 0: 0$
(b) (iii) zero unless $x = n\pi$

3. (a) $\frac{1}{\sqrt{2}}(1+x-\frac{x^2}{2!}+\frac{x^3}{3!}+\dots)$ (b) binomial!, $\frac{1}{(a^2+b^2x^2)^3} = a^{-6}(1-\frac{3b^2x^2}{a^2}+\frac{6b^4x^4}{a^4}+\dots)$
(c) $1+\frac{1}{2}x^2+\frac{3}{8}x^4$

4. $(a, b, c) \sigma (a, -b, -c) \sigma (-a, b, c) \sigma (-a, -b, c) f=3$ Maxima

5. (a) $y = (x^3+x^2+x) \sec x$ (b) $y = (\frac{1}{2} \cos 2x + x^2/2 + 3/2)^{-1}$
(c) $y = (1-\cos x) e^{-x^2/2}$ (d) $x = (1-2(y/x)) - (y/x)^2)^{-1/2}$

6. (a) (i) A_{ji} (ii) $A_{ik}B_{kj}$ (iii) A_{ii} (iv) $A_{ik}B_{ki}$ (v) $A_{ik}B_{kl}C_{lj}$ (vi) $A_{ik}B_{lk}C_{jl}$ (b) !!!

7. (b) (i) 2 (ii) 1 or 3 (iii) 5/21 (c) Most probable = 4 = expected

8. (a) - (b) - (c) - (d) $x = k(u \wedge v)$

9. (a) (2, -2, -1) (b) $\Gamma \cdot (a \wedge b + b \wedge c + c \wedge a) = a \cdot (b \wedge c)$ (c) $x+2z=4$

10. (a) (0, 0, πa^2) (b) $\pi a^2 y$ (c) $\pi^2 a^3/3$ (d) zero and $1+x+xy$

11. (a) (i) $Re = -2$ $|m|=0$ (ii) $Re \in \text{ln} r$ $|m| = \theta + \pi/2 + 2n\pi$ (b) (i) 0 to $\pi/2$ (ii) $\pi/2$ to π

(c) (i) $b > 0$ (ii) No soln (?)

12. $x^2 = \pi/3 + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx \quad x = \sum_{n=1}^{\infty} -2 \frac{(-1)^n}{n} \sin nx$

III| : (b) $\mu = 1/u^2 - \frac{1}{x+y} + e^{y/(x+y)} = c$

2. -

3. -

4. (a) (2, -3) saddle (b) couldn't be bothered

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5. (a) (i) 0 (ii) $(\beta e^\pi \cos \beta + \beta - e^\pi \sin \beta)/(1+\beta^2)$ (iii) $\frac{r^2 h}{2} \rightarrow -\frac{r^2}{4} + \frac{1}{4}$

6. (a) (i) 1 (ii) zero (b) $-z_0^2 r^2 \pi$ (c) $\nabla \phi = (0, z^2, 2yz)$

7. (d) couldn't be bothered

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8. $z = \alpha \sin \pi x \sin \pi y \cos \sqrt{2}\pi z t$; max speed = $\sqrt{2}\pi \alpha$ at $x=y=1/2$

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9. (b) $b_0 = 1, b_1 = 1, b_2 = 5/2, b_3 = 11/6$ (c) $c_0 = c_1 = c_2 = 1, c_3 = 2/3$ (\therefore wrong)

9. (b) $b_0 = 1, b_1 = 1, b_2 = 5/2, b_3 = 11/6$ (c) $c_0 = c_1 = c_2 = 1, c_3 = 2/3$ (\therefore wrong)

10. (b) $\lambda = -3, -6 \text{ or } 12$ (c) $\frac{1}{\sqrt{3}}(1, -1, 1), \frac{1}{\sqrt{2}}(1, 0, -1), \frac{1}{\sqrt{6}}(1, 2, 1)$

11. (a) $y = A_1 e^{-2x/\lambda} + A_2 e^x + e^{-x}/(2\lambda-4)$

(b) $y = A_1 e^{-2x/\lambda} + A_2 e^x + x e^x / (\lambda+2)$

CF is $y = (Ax+B)e^x$; for $\lambda=0$ need one boundary condition

12. (a) continuous, not differentiable (i) continuous, differentiable

(b) $1/\lambda^2$.

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I/1 (a) $2\hat{i} - \hat{j} - 2\hat{k} + \lambda(3\hat{i} + \hat{j} - 3\hat{k})$ (b) $\pm \frac{1}{\sqrt{2}} (\hat{i} + \hat{k})$ (c) $\frac{1}{2}\sqrt{154}$ (d) $\frac{5}{\sqrt{154}}$

2. $f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi^2(2n-1)^2} \cos(2n-1)x$

3. (a) $\lambda = \cos\theta \pm i\sin\theta$ (b) $\frac{1}{\sqrt{2}} (\pm 1)$ (c) $\lambda = 1 \pm i\sin 2\theta, \frac{1}{\sqrt{2}} (\pm 1)$

4. -

5. (a) touch at $x=1$

6. (a) $\frac{8}{3}abc(a^2+b^2+c^2)$ (b) $4\pi abc$

7. (a)(i) $\begin{pmatrix} 2 & -10 & 15 \\ -6 & -3 & -6 \end{pmatrix}$ (ii) $\begin{pmatrix} 6 & 12 \\ 2 & 4 \end{pmatrix}$ (iii) 48 (iv) $\begin{pmatrix} -74 & 0 & 74 \\ -74 & 0 & 74 \\ -74 & 0 & 74 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1/2 \\ 1 & -3/2 & 1/2 \end{pmatrix}$

8. -

9. (a) $x - x^2/2 + x^3/3 - x^4/4 + \dots$ (b) $x/a - x^3/2a^3 + 3x^5/8a^5 - \dots$

(c) $e^{-ax}(1 + 2ax + (2a^2 - 1)x^2 + \dots)$ (d) $-x - 5x^2/2 - x^3/3$

10. (a)(i) $a_{ij} b_{ji}$ (ii) $a_{ik} b_{kl} c_{lj}$ (iii) $\cap ?!$ or $\delta_{ii}; \text{Tr}(ABA^{-1}) = \text{Tr}(B)$

11. (a)(i) $\text{Re} = -1, \text{Im} = 0$ (ii) $\text{Re} = 0, \text{Im} = \frac{\phi}{2} + 2n\pi$ (b) $|e^{i\theta}|$ with $\theta = 2n\pi/3$

(b)(ii) $2e^{i\theta}$ with $\theta = 2n\pi/3$ (c) $-1 \leq b \leq +1$

12. (b) Mean = 0 $V\omega = 1/6$ (c)(ii) 3

II/1: (a) $y^2 = 3x^2 - 11$ (b) $y = (1 - \cos 2x)/4 \sin x$ (c) $e^{yx}(1 + 2ky^2) = 2kx^2$

2. n^{th} derivative of $x(x+1)e^{2x}$ is $2^n - 2 e^{2x} (4x^2 + 4x + 4nx + n(n+1))$

3. ?

4. (a) $y = Ae^{-x} + Be^{-2x} + \frac{1}{2}x - \frac{3}{4} - xe^{-x}$ (b) $y = \frac{(Ax+B+x^2)e^{-x}}{x^2 - 4x + 6}$

5. (a)(i) $e+1$ (ii) $e+1$ (b) zero (c) $f = ye^x + y^2 z^3$

6. (a) - (b) $\frac{1}{\sqrt{5}} = \sqrt{5} - 1$ Area = $(\sqrt{5} + 1)\pi$

7. (a)(i) 19 (ii) 30 (iii) zero (b) $\lambda = 1 \pm 6, \frac{1}{\sqrt{14}}(3, -2, 1)$ and $\frac{1}{7}(6, 3, 2)$ (c) -2

8. (a) $8/3$ (b) $8\pi/3$

9. $\frac{dU}{dT} = (an^2 e^{-an/VRT})/V(V-nb); U = TC_V + \text{constant}$

10. $(\partial U/\partial V)_T = (an^2 e^{-an/VRT})/V(V-nb)$

11. (a) $(5/7)^{n-1}(2/7)$ (b) $n^{2/7}$ (ii) $2^{2/7}$ (iii) $1/21$ (iv) $11/21$ (v) $1/6$

12. (a) (i) Yes: 2 (ii) Yes: 0 (iii) Yes: 1 (b) - (c) Convergent

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I/I: (i) $\underline{c} \cdot \underline{b} = \underline{a} \cdot \underline{b}$ (ii) $\underline{c} \cdot (\underline{a} \wedge \underline{b} + \underline{b} \wedge \underline{c} + \underline{c} \wedge \underline{a}) = \underline{a} \cdot \underline{b} \wedge \underline{c}$
 (iii) $\underline{c} \cdot (\underline{b} \wedge \underline{c} - \underline{a} \wedge \underline{c}) = \underline{a} \cdot (\underline{b} \wedge \underline{c})$

2. $x^2 = \pi^2/3 + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx$

3. (a) Max at $x = 1/3$; min at $x = 3$

(b) (i) $y = -2x/11 + 24/11$ (ii) $y = 2x - 2$

4. Max 2 at $\pm(0, \sqrt{2}, \sqrt{2})$; Min $\frac{2}{\sqrt{3}}$ at $\pm\left(\frac{2\sqrt{2}}{3}, \frac{\sqrt{2}}{3}, -\frac{\sqrt{2}}{3}\right)$

5. (a) $\sqrt{\pi}/2, \sqrt{\pi}/4, \frac{-1}{2}e^{-x^2} + C, \frac{-(x^2+1)}{2}e^{-x^2} + C$ (b) $\frac{1}{2} - \frac{1}{2}e^{-1} - \frac{1}{2}e^{-4}$

6. (a) $A^{-1} = \frac{1}{5} \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$ (b) $\lambda = 5, \mu = 1$

7. (a) (i) $y+1 = k(x-1)^{1/2}/(x+1)^{1/2}$ (ii) $y = \frac{1}{2}(\sec x + \tan x) + C(\sec x + \tan x)^{-1}$

(b) $y^{-1} = \frac{1}{2}(x^2 + x + 1/2) + \frac{3}{4}e^{2x}$

8. (a) $\text{Tr} = 2, \det = 1$ (b) - (c) -

9. F not, G is; all integrals zero except F along (i), which gives 1

10. $f = c(t+b)^{-1/2}; \theta(x,t) = \frac{1}{2(t+1/4)^{1/2}} \left[e^{-(x-2)^2/(4t+1)} - e^{-(x+2)^2/(4t+1)} \right]$

11. (a) $\ln \omega = \ln \left(\frac{\sin \phi}{1+\cos \phi} \right) + i\pi/2 \pm 2i\pi n, n=0,1,2,\dots$
 (b) $\ln \omega = \ln \left(\frac{\sin \phi}{1+\cos \phi} \right) + i\pi/2 \pm 2i\pi n$
 (c) $\ln \omega = \ln \left(\frac{\sin \phi}{1+\cos \phi} \right) + i\pi/2 \pm 2i\pi n$

12. $f(x,y) = \sum_{m=0}^{\infty} (8 \sinh(2m+1)x \sin(2m+1)y) / ((2m+1)\pi(e^{(2m+1)\pi} - e^{-(2m+1)\pi}))$

IV I: $\underline{c} \cdot \underline{a} = \underline{c} \cdot \underline{a} / \lambda$ holds, but after that all I can get is a complicated curve

2. (a) $36\pi a^3/a$ (b) $\beta/(\beta^2+x^2)$ (c) $(\beta^2-x^2)/(\beta^2+x^2)$

3. (a) $8\pi\mu a^3/3$ (b) $\lambda a^5/15$

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4. $P_r = \binom{n}{r} p^r q^{n-r}; \mu = np, \sigma = \sqrt{npq};$ Expected correct $= 15$

Prob is $> 3\sqrt{5}/2\sqrt{45} \approx > 1/2.$

5. Unique soln unless $\lambda = 1 \text{ or } 2;$ when $\lambda = 1 \Rightarrow \lambda = \mu \text{ and } \frac{z}{y} = 1 - \frac{1}{2}\mu;$
 when $\lambda = 1$ no solns; when $\lambda = 2 \Rightarrow y = 1, x = \text{anything}, z = -x/2, \mu = 0.$

6. $\nabla \cdot \underline{B} = 0; 2\pi, 2\pi, 2\pi, 2\pi, e^{-3x}+$

7. (a) $y = A(e^{-2x} - \frac{2}{3}e^{-3x}) + \frac{1}{6}x - 5$ (b) $y = A \cos \sqrt{2}x + B \sin \sqrt{2}x$

7. (a) not necessarily antisymm and cannot be orthogonal

8. (a) not necessarily antisymm and cannot be orthogonal

(b) $\lambda = -4, 3, 8; \frac{1}{2}(\sqrt{3}, 0, 1), (0, 1, 0), \frac{1}{2}(1, 0, -\sqrt{3})$

9. (i) $1 + x/2 + x^2/12$ (ii) $1 + 2x + 2x^2$ (iii) $x^2/2 + x^4/12 + x^6/45$

10. (i) all α (ii) $\alpha \neq \pm 1$ (iii) $\alpha > 0$ (iv) all α

11. $\frac{\partial^2 v}{\partial t^2} - \frac{\partial v}{\partial s} = 0$

12. saddles at: $(0,0), (\pm 1/\sqrt{2}, \pm \sqrt{2});$ Max at $(0, \pm \sqrt{2});$ Min at $(\pm \frac{1}{\sqrt{2}}, 0)$

I/II (a) $z^* = 2e^{-i\phi}, z^{-1} = \frac{1}{2}e^{-i\phi}, -z = -2e^{i\phi}$ (b) $\cos 2\theta = 2\cos^2 \theta - 1$
 $\sin 2\theta = 2\cos \theta \sin \theta$

2003

$$\text{I/1: } (2, 1, -1) = -\frac{1}{3}(-1, 1, 0) + \frac{2}{3}(0, 2, 1) + \frac{5}{3}(1, 0, -1)$$

$$2. (a) i) 3, 4, 2/3 ii) e^{-x} (b) (i) 1/3 (ii) \pi^2/24$$

$$3. (a) x=0 \text{ saddle}; x=1 \text{ max}; x=-1 \text{ min} (c) \infty, 2/27$$

$$4. (x+x^3) \ln(1+x^2) - \frac{2}{3}x^3 + C (i) \ln(\ln x) + C (ii) \frac{\cos^3 x}{3} - \cos^{-1} x + C$$

5. (c) $n=3$ except for (iii), where $n=\text{anything}$

$$6. u = (3 \sin x)e^{-t} - (\sin 3x)e^{-9t}$$

$$7. (i) \infty (ii) \frac{1}{3}(\lambda - \mu - 1) (iii) \mu = 0 \lambda = 1$$

$$8. (a) (i) 1+x+x^2/2 (ii) x^3-x^5/2+13x^7/120 (iii) e^x + xe^x + x^2e^x$$

$$(b) \frac{1}{a_0} - \frac{a_1}{a_0^2}x + \left(\frac{2a_1^2}{a_0^3} - \frac{2a_2}{a_0^2} \right) \frac{x^2}{2}$$

$$9. (b) \sinh x = x + x^3/3! + x^5/5! + \dots \cosh x = 1 + x^2/2! + x^4/4! + \dots$$

$$(c) \frac{1}{2} \cosh \frac{\pi}{2}$$

$$10. (b) 5 \cdot 2$$

$$11. (a) (i) y = e^{kt \tan^{-1} x + kc} (ii) y = Ae^{x^3/3} - 1 (b) \cos \frac{y}{x} = \frac{e^{(1/x-1)}}{\sqrt{2}}$$

12. —

$$\text{II/1: (a) } \underline{\text{L}} \cdot \underline{(-1, 2, 2)} = \frac{2}{3} \text{ shortest distance} = \frac{2}{3} (b) \frac{1}{3}(2, -1, -2) \text{ and } (6, 2, 5)$$

$$2. \sin nx = \sum_{n=1}^{\infty} \frac{2n}{\pi} (-1)^n \frac{\sin n\pi}{n^2-n^2} \sin nx$$

$$3. (a) (a, b) \text{ is } (0 \neq 2) \text{ or } (\pm 2, 0) (b) \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ and } \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}; \lambda = 156.$$

$$4. (b) 3/4 (c) -\pi r^2$$

$$5. (a) a_n / (a_n + b(N-n)) (b) (i) A = k (ii) e^{-2k} (iii) \frac{1}{k}$$

$$6. \text{The inverse is } \begin{pmatrix} 2 & -5 & 16 \\ -1 & 3 & -10 \end{pmatrix}$$

$$7. \text{grad} = \left(\frac{2x}{a^2}, \frac{2y}{b^2}, 2z \right); \text{final integral} = \frac{a^2-b^2}{3} \left[\left(1 - \frac{1}{a^2} - \frac{1}{b^2} \right)^{3/2} - \left(1 - \frac{1}{b^2} \right)^{3/2} - \left(1 - \frac{1}{a^2} \right)^{3/2} + 1 \right]$$

$$8. (a) (i) n=2 (ii) \det P = 0 (b) -$$

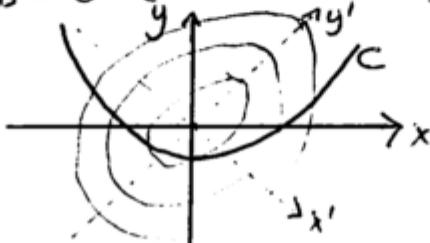
$$9. (b) C_P - C_V = Nk$$

$$10. (a) \frac{\partial^2 r}{\partial x^2} = y^2(x^2+y^2)^{-3/2}; \frac{\partial^2 r}{\partial y^2} = x^2(x^2+y^2)^{-3/2}; \frac{\partial^2 r}{\partial x \partial y} = \frac{\partial^2 r}{\partial y \partial x} = -xy(x^2+y^2)^{-3/2}$$

(b) —

$$11. a = -1 \quad b = 0 \quad c = 1 \quad d = 1; \quad y = (x-1)e^x - \sin x + (1+x)\cos x$$

12. work here



put $x' = x-y$, $y' = x+y$
contours of u follow

ellipses
C is as shown. Stationary
points occur when C is
parallel to contours

2004

I/1: (i) $x+z=4$ (ii) 8

2. (i) conv (ii) div (iii) conv (iv) conv

3. (iii) Both equal 1/2

4. (a) $h \left\{ \frac{(b+1)^2}{(a+1)^2} \frac{a}{b} \right\}$ (b) $\tan x - x + c$ (c) $I_3 < I_1 < I_2$

5. (a) $k=1$ mean = 1 variance = 1 (b) (i) $M=1/2$ (ii) mean = 0 (iii) $Var = \frac{3}{2}$ (iv) e^{-1}

6. (i) $x=1/2$ $y=1/2$ $z=-1$ (ii) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ (iii) $x = k \begin{pmatrix} -3 \\ 1 \\ 1/2 \end{pmatrix}$ $k = \text{any scalar}$

7. (b) (i) 3 (ii) $1+4(a_1^2+a_2^2+a_3^2)$ (iv) ~~zero~~

8. (i) 1 and 1 (ii) 1 and 1 (iii) $\nabla F = x^3 y^2 z$ F is not conservative

9. (a) (i) $\cos 2 + i \sin 2$ (ii) $(1+2i)\pi$ (iii) $(\frac{1}{2}+2i)\pi$ (iv) $\cos(6\pi) + i \sin(6\pi)$
(v) $e^{-(1/2+2i)\pi}$ (c) (i) ~~zero~~ (ii) $\ln 5/4$

10. (a) $\cos 5\theta = 16\cos^5 \theta - 20\cos^3 \theta + 5\cos \theta$ (b) (i) an ellipse (ii) circle $(3+2i)$,
radius 8 (iii) circle $(3i-2)$, radius 8 (c) (i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ii) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

11. (a) $e^y = k(1-x^2)^{1/2}$ (b) $y^{-2} = 3e^{-2x} + 1$ (c) $y = c \sin x - \sin^2 x - 1$

12. Mass = $\pi \rho_0 h a^2 / 2$ $a = 7h/30$

II/1. -

2. $3 + \sum_{m=1}^{\infty} \frac{12}{(2m-1)\pi} \sin \frac{(2m-1)\pi \lambda}{5}$; converges to 3 in all cases

3. (b) (ii) $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sigma \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

4. (a) 4/3 (b) $4 \sin 4\theta / \sin \theta, 16, (1,1)$ ~~circle~~

5. (i) $x = 1 - 193/256$ so prob = $193/512$ (ii) prob = $[1 - 2p(1-p)]^k 2p(1-p)$

6. $\phi = \sum_{n=1}^{\infty} B_n \sin nx \sin \sqrt{25-n^2} y$ $\phi = 0$

7. (i) $\underline{u} = ((3x^2+y)/y, (x+z)(y, 1))$

8. (a) (i) $x+x^2/2+5x^3/6$ (ii) $1-x+x^2$ (iii) $-x^2/2-x^4/12-x^6/45$

(b) (i) $x_0 = a_0 b_0$ $c_1 = a_0 b_1 + a_1 b_0$ $c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$ (ii) $c_i = \sum_{j=0}^i a_j b_{i-j}$

9. (b) (i) $p k + p(p-1)k^2/100 + \dots$ (ii) $k p + m q + \frac{1}{200} (k^2 p(p-1) + 2kmpq + m^2 q(q-1))$

10. (b) (ii) $k_2 = 0.482$ $k_4 = 0.464$ (iii) 1.20

11. (i) $\alpha=3 \beta=2$ and $\alpha=0 \beta=1$ (ii) $y = -1/x^2 + 1/x$ $y = \sin(\ln x)$

12. (a) $a = (4V/3)^{1/3} = 2b = 2c$ cost = $(36V)^{2/3}/2$ ~~circle~~

(b) $(-\pi/2, -\pi/2)$ max; $(3\pi/2, 3\pi/2)$ min; $(-\pi/2, \pi/2), (\pi/2, -\pi/2)$ saddles