2015

Paper I

A1. (a) $(x+1)(x-1)^2$; (b) $\frac{1}{4}\left(\frac{3-x}{(x-1)^2}+\frac{1}{x+1}\right)$. 2. (a) $m=e$; (b) $(1,e)$.

3. (a) circle centre $(1,0)$ radius 2. 4. (a) $-\operatorname{cosec}^2 x$. 5. (a) $\frac{x^2}{2}\sin x^2+\frac{1}{2}\cos x^2+c$; (b) $\frac{\pi}{4}-\frac{1}{2}$.

6. (a) $y=1, x=\frac{\sqrt{3}}{2},-\frac{\sqrt{3}}{2},0$; (b) $-3\cos 3t/\sin t$. 7. $1-\frac{3x}{2}-\frac{9x^2}{8}$; $\frac{-1}{3}<x\le\frac{1}{3}$.

8. $x=\pm\frac{\pi}{2},\pm\frac{\pi}{4},\pm\frac{3\pi}{4}$. 9. (a) $-\sqrt{6}$; (b) $y=\frac{x}{\sqrt{6}}+\frac{1}{2}$. 10. (a) 500500; (b) 4094.

B11. (a)(i) -1, $-i$; (b) $\dot z=(\dot r+ir\dot\theta)e^{i\theta}, \ddot z=(\ddot r-r\dot\theta^2+i(2\dot r\dot\theta+r\ddot\theta))e^{i\theta}$; (c)(i) circle centre $(0,0)$, radius 2; (ii) $\dot r=\ddot r=0, \dot z=2i\dot\theta e^{i\theta}, \ddot z=(-2\dot\theta^2+2i\ddot\theta)e^{i\theta}$;

(iii) radial = $\ddot r-r\dot\theta^2$, transverse = $2\dot r\dot\theta+r\ddot\theta$.

12. (a)(i) exact, $f=e^{x+y}+c$; (ii) not exact; (iii) exact, $f=x^2y^3z^4+c$;

(b) $g(2m\pi,0)=0$ minima, $g((2m+1)\pi,0)=2$ saddles.

13. (b) $u(x,t)=(t+a)^{\frac{-1}{2}}e^{\frac{-(x+b)^2}{4\lambda(t+a)}}$; (c) $(4\lambda t+1)^{\frac{-1}{2}}\left(e^{\frac{-(x+1)^2}{4\lambda t+1}}+e^{\frac{-(x-1)^2}{4\lambda t+1}}\right)$, $t>\frac{-1}{4\lambda}$.

14. (a) $a-a\cos t$, $a\sin t$, $\frac{\sin t}{1-\cos t}$; (c) $a\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$; (d) $3\pi a^2$; (e) $8a$.

15. (b) $\cosh\sqrt{2}+\frac{(x-2)}{2\sqrt{2}}\sinh\sqrt{2}+\frac{(x-2)^2}{16}\left(\cosh\sqrt{2}-\frac{1}{\sqrt{2}}\sinh\sqrt{2}\right)$; (c)(i) $x-2x^2+\frac{17x^3}{6}$;

(ii) $1+\frac{x^2}{3}-\frac{19x^4}{120}$.

16. (a)(i) 3/7, 4/7, 3/7; (ii) 2/7; (iii) 1/2, 1/2; (iv) 6/7; (v) 12/7; (b) $\frac{N_B}{N}$.

17. (a) $\frac{b^3}{3}$; (b)(i) $\sin^{-1}\left(\frac{x}{2}\right)+c$; (ii) $\ln\sqrt{\left(\frac{x}{2}+1\right)^2+1}-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}+1\right)+c$;

(iii) $F(k)=\pi^k-k(k-1)F(k-2)$; $F(5)=\pi^5-20\pi^3+120\pi$.

18. (d) $\lambda=1, \mathrm{v}=(1\quad 1\quad 1)^{\mathrm{T}}$, $\lambda=e^{\frac{2i\pi}{3}}, \mathrm{v}=\left(1\quad e^{\frac{4i\pi}{3}}\quad e^{\frac{2i\pi}{3}}\right)^{\mathrm{T}}$, $\lambda=e^{\frac{4i\pi}{3}}, \mathrm{v}=\left(1\quad e^{\frac{2i\pi}{3}}\quad e^{\frac{4i\pi}{3}}\right)^{\mathrm{T}}$;

(e) $\lambda=1, \mathrm{v}=(1\quad 1\quad 1)^{\mathrm{T}}$, $\lambda=e^{\frac{2i\pi}{3}}, \mathrm{v}=\left(1\quad e^{\frac{2i\pi}{3}}\quad e^{\frac{4i\pi}{3}}\right)^{\mathrm{T}}$, $\lambda=e^{\frac{4i\pi}{3}}, \mathrm{v}=\left(1\quad e^{\frac{4i\pi}{3}}\quad e^{\frac{2i\pi}{3}}\right)^{\mathrm{T}}$;

(f) $\frac{1}{2}(\mathrm{A}+\mathrm{A}^{\mathrm{T}})$; (g) $1,-\frac{1}{2},-\frac{1}{2}$.

19. (a) $\frac{3}{16}\left[4N\left(\frac{-1}{3}\right)^{N+1}-\left(\frac{-1}{3}\right)^N+1\right]$, $\frac{3}{16}$.

20. (c) $-\alpha\rho\hat\rho+2\alpha z k$; (e) $\pm\alpha RA$ (depending on which way round the edge one goes).

Paper II

A1. (a) $\hat n=\frac{1}{\sqrt{30}}(-5\quad 1\quad 2)^{\mathrm{T}}, \frac{2}{\sqrt{30}}$. 2. (a) $-2, 1\pm i\sqrt{3}$. 3. x^2, minimum.

4. $\lambda=-1, \mathrm{v}=\frac{1}{\sqrt{2}}(1\quad -1)^{\mathrm{T}}$, $\lambda=3, \mathrm{v}=\frac{1}{\sqrt{2}}(1\quad 1)^{\mathrm{T}}$.

5. $F=\begin{pmatrix}\cos y^5\sinh z & -5y^4x\sin y^5\sinh z & x\cos y^5\cosh z\end{pmatrix}^{\mathrm{T}}$, $\operatorname{curl}F=0$. 6. (a) $2r$; (b) $3a$.

7. $y=x^2$. 8. $\frac{4}{5}\pi R^5$. 9. $\sin x+2\sin 2x$. 10. (a) $\frac{1}{2^{10}}$; (b) $1-\frac{11}{2^{10}}$.

B11. (a) $\lambda=1,\mu=-3,\mathrm{p}=\begin{pmatrix}3 & 0 & 1\end{pmatrix}^{\mathrm{T}}$; (b)(i) the point i; (ii) the line $(t,-1-t,-1-t)$; (iii) no points.

12. (b) $a^2\pi\sin\frac{\beta}{2}$ from the centre along the line through the centre of mass.

13. (a) ***F*** is not; ***G*** is, $\phi=-x^2y\cosh z+c$; (b) both $\cosh 1$; (c) $3\cosh 1-3\sinh 1$, $\cosh 1$.

14. (b) $\frac{2}{T_2^2}$; (c) $\frac{t}{T_1}$ for $0<t<T_1$; (d) $\frac{t^2}{T_2^2}$ for $0<t<T_2$; (e) $\frac{t^3}{T^3}$ for $0<t<T$;

(f) $R(t)=\frac{t}{T}+\frac{t^2}{T^2}-\frac{t^3}{T^3}$ for $0<t<T$; (f) $\left(\frac{T}{3},\frac{11}{27}\right)$.

15. (a)(i) no solutions; (ii) $f=e^{-2t}+2e^{-6t}-3e^{-4t}$; (b) $y=(1-2x+2x^2)e^{-x}$, $y=5e^{-2}$.

16. (a) $\frac{1}{P}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)$ fn of y only; (i) $y=kx^{-1/2}-x$; (ii) $y+y^2\tan x=k$;

(b) $\ln\sqrt{y^2+2xy+2x^2}-2\tan^{-1}\left(\frac{y}{x}+1\right)=k$.

17. You might get different versions depending on how you form your "eigenvector equation":

(c) $\begin{pmatrix}2k & -k & 0\\ -k & 2k & -k\\ 0 & -k & 2k\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\omega^2 m\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$; (d) $\lambda=2k,\mathrm{v}=\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix}^{\mathrm{T}}$,

$\lambda=k(2+\sqrt{2}),\mathrm{v}=\frac{1}{2}\begin{pmatrix}1 & -\sqrt{2} & 1\end{pmatrix}^{\mathrm{T}}$, $\lambda=k(2-\sqrt{2}),\mathrm{v}=\frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\end{pmatrix}^{\mathrm{T}}$; (f) x_1 and x_3 are in phase, x_2 is opposite in phase and $\sqrt{2}$ bigger in magnitude.

18. (a) $\omega=\frac{2\pi}{T}$; (c) $a_n=0$ if n is odd, $a_n=\frac{2V_0}{\pi(1-n^2)}$ if n is even, $b_1=V_0/2$, $b_n=0$ otherwise.

19. (b) $f=\ln n$; (c) $\sum_{i=1}^{n}ae^{-by_i}=1$, $\sum_{i=1}^{n}y_iae^{-by_i}=Y$.

20. (a) $u_s=\frac{G}{n^2\pi^2\nu}\sin n\pi x$; (b) $\frac{\partial\tilde{u}}{\partial t}=\nu\frac{\partial^2\tilde{u}}{\partial x^2}$, $\tilde{u}(0,t)=0$, $\tilde{u}(1,t)=0$, $\tilde{u}(x,0)=\frac{-G}{n^2\pi^2\nu}\sin n\pi x$;

(c) $\tilde{u}=\frac{-G}{n^2\pi^2\nu}e^{-n^2\pi^2\nu t}\sin n\pi x$;

(d) $u(x,t)=\frac{G}{n^2\pi^2\nu}\sin n\pi x\left(1-e^{-n^2\pi^2\nu t}\right)$, $Q(t)=\frac{G\left(1-e^{-n^2\pi^2\nu t}\right)(1-\cos n\pi)}{n^3\pi^3\nu}$.

2016

Paper I

A1. (a) $(a+b)(a^2-ab+b^2)$; (b) $x=2$. 2. (a) $-\frac{1}{2}(x-1)^2+\frac{1}{2}$; (b) $\frac{\frac{3}{2}}{x-1}-\frac{\frac{1}{2}}{x+1}$.

3. (a) 10^8-4; (b) 0. 4. $\left[\frac{4\pi}{3},\frac{5\pi}{3}\right]\pm 2n\pi$. 5. (a) $x=\pm\frac{1}{\sqrt{2}}$. 6. (a) $y=(2-x)^{-1}$.

7. (a) $-\frac{1}{3}\cot(3x+1)+c$; (b) $\frac{(n-2)^{n+1}}{n+1}-\frac{(-2)^{n+1}}{n+1}$. 8. $-\sin x\,(\ln 2+1)(2e)^{\cos x}$. 9. -. 10. $0\le k\le\frac{4}{3}$.

B11. (a) $y=0$; (b) $z=i$; (c) $b=\frac{(\lambda^2+1)i}{(\lambda^2-1)}$, $c=1$, centre is $\left(0,-\frac{(\lambda^2+1)}{(\lambda^2-1)}\right)$, radius is $\frac{2\lambda}{|\lambda^2-1|}$;

(d)(i) $|b|^2=c$; (ii) $|b|^2>c$; (iii) impossible.

12. (b)(i) Min at (0,0), saddles at $(\pm1,\pm1)$.

13. (a) $\boldsymbol{\nabla}.\mathbf{v_1}=0$, $\boldsymbol{\nabla}.\mathbf{v_2}=10y$; (b) $I_1=\frac{5\pi}{2}bc(a_zb-a_x)+2bc\left(a_x-a_y\right)$, $I_2=b^3+25\pi^2c^2b/4$;

(c) $\mathbf{v_1}$ is not, $\mathbf{v_2}$ is conservative; (d) $\phi=\mathrm{x}^2y+y^3+z^2y+c$.

14. (b)(i) $\lambda=1,(0\quad 0\quad 1)^T,\lambda=2,\frac{1}{\sqrt{2}}(1\quad 1\quad 0)^T,\lambda=4,\frac{1}{\sqrt{2}}(1\quad -1\quad 0)^T$; (ii) $\frac{1}{8}\begin{pmatrix}3&1&0\\1&3&0\\0&0&8\end{pmatrix}$.

15. (a)(i) $\frac{1}{2}\ln 2+\frac{x}{4}(\ln 2+1)+\frac{x^2}{8}\left(\ln 2+\frac{1}{2}\right)$; (ii) $x-\frac{x^3}{3}+\frac{x^5}{5}$; (iii) $\frac{x^2}{2}-\frac{x^4}{12}+\frac{x^6}{45}$;

(b)(i) $\frac{d^2g}{df^2}=-\frac{d^2f}{dg^2}/\left(\frac{df}{dg}\right)^3,\frac{d^3g}{df^3}=3\left(\frac{d^2f}{dg^2}\right)^2/\left(\frac{df}{dg}\right)^5-\frac{d^3f}{dg^3}/\left(\frac{df}{dg}\right)^4$;

(ii) $b_1=1/a_1$, $b_2=-a_2/a_1^3$, $b_3=2a_2^2/a_1^5-\ a_3/a_1^4$.

16. (a)(i) $(R-r)/R$; (v) $r^2/2R$; (vi) $r^3/3R\ -r^4/4R^2$; (b)(i) $(r-t)/R$; (ii) $(R-r)/r$; (iii) t/R.

17. (a)(i) $y=a_1x+a_0$; (ii) $\sum_{i=0}^{n-1}a_ix^i$; (b)(i) $A_0+A_1e^x+A_2e^{-x}+A_3\cos x+A_4\sin x$; (ii) $-\frac{1}{2}x^2$;

(iv) $y=1+\frac{\cosh x}{2}-\frac{1}{2}\cos x-\frac{x^2}{2}$.

18. (a)(iii) $I=\frac{1}{\alpha}$; (b)(ii) $\frac{(m-2)!(n-2)!}{(m+n-3)!}$.

19. (a) & (b) $\begin{pmatrix}\tan\theta\cos\phi & \tan\theta\sin\phi & 1\\ W\sec^2\theta\cos\phi & W\sec^2\theta\sin\phi & 0\\ -\mathrm{W}\tan\theta\sin\phi & \mathrm{W}\tan\theta\cos\phi & 0\end{pmatrix}$; (c) $\frac{1}{W}\begin{pmatrix}0 & \cos^2\theta\cos\phi & -\cot\theta\sin\phi\\ 0 & \cos^2\theta\sin\phi & -\cot\theta\cos\phi\\ W & \cos^2\theta\tan\theta & 0\end{pmatrix}$.

20. (a)(ii) Yes; (iii) $(y^2\quad z^2\quad x^2)^T$; (b)(ii) $\pi a^3b/4$; (c) $\pi a^3b/4$.

Paper II

A1. $x\ge1,y=0$. 2. $\boldsymbol{r_0}=(1\quad 2\quad -1)^T,t=(4\quad 3\quad 2)^T$. 3. $\frac{\partial f}{\partial r}=\cos2\theta\frac{\partial f}{\partial x}+\sin2\theta\frac{\partial f}{\partial y}$,

$\frac{\partial f}{\partial\theta}=-2\mathrm{r}\sin2\theta\frac{\partial f}{\partial x}+2\mathrm{r}\cos2\theta\frac{\partial f}{\partial y}$. 4(i) $\left(\frac{35}{36}\right)^2\frac{1}{36}$; (ii) $\frac{36}{71}$. 5. (a) Yes; (b) No. 6. (a) Zero; (b) $\sin\frac{\pi}{8}\cos\frac{\pi}{4}$.

7. (a) $\sin x+\sin2x+\sin3x$; (b) $\cos x+2\cos2x+3\cos3x$. 8. $y=xe^x$.

9. $\frac{4}{3}\pi(b^3-a^3)+4\pi\left(\frac{1}{c}-\frac{1}{d}\right)$. 10. (a) xy; (b) saddle.

B11. (a)(i) $\mathbf{r}.(\mathbf{a}\wedge\mathbf{b})=0$; (ii) $\mathbf{r}.(\mathbf{a}\wedge\mathbf{b})>0$; (c) $\mathbf{r}.(\mathbf{a}\wedge\mathbf{b})>0$, $\mathbf{r}.(\mathbf{b}\wedge\mathbf{c})>0$, $\mathbf{r}.(\mathbf{c}\wedge\mathbf{a})>0$,

$\mathbf{r}.(\mathbf{a}\wedge\mathbf{b}+\mathbf{b}\wedge\mathbf{c}+\mathbf{c}\wedge\mathbf{a})<\mathbf{a}.(\mathbf{b}\wedge\mathbf{c})$; (d) $\sqrt{6}$.

12. (a) $\left(r,\theta,a(1-r^2/b^2)\right)$; (b) $\alpha ab^2/4$; (d) $\alpha b^2/2$;

(e) $\mathbf{S}_{ABCA}=\left(\left(\frac{2ab}{3}\right)\sin\alpha\quad\left(\frac{2ab}{3}\right)(1-\cos\alpha)\quad\alpha b^2/2\right)^T$.

13. (b)(i) $t=0$ to infinity; (ii) $3a/2$; (c) $y^3+x^3-3ayx=0$, $x=y$.

14. (a) $y=\frac{1}{3}(a^2+x^2)+c(a^2+x^2)^{-\frac{1}{2}}$; (b)(i) $|x+y-2|=Ae^{-\frac{1}{2}(x+3y)}$; (ii) Yes, $y=2-x$;

(c)(ii) $x=\frac{B|1-p^2|}{p^2}$; (iii) $y^2=2Bx+B^2$.

15. (b) $x = \frac{t-s}{t-1}, y = \frac{s+t-2}{t-1}, z = -1$

$t \neq 1$ one solution, $t = 1, s = 1$ infinitely many solutions, $t = 1, s = 6$ no solutions.

16. (a)(i) $0{:}\frac{1}{10}, 1{:}\frac{3}{5}, 2{:}\frac{3}{10}$; (ii) $\mu = \frac{6}{5}, \sigma = \frac{3}{5}$; (iii) $\frac{3}{5}$; (iv) $\frac{7}{10}$; (b)(i) $\frac{13}{30}$; (ii) $\frac{12}{13}$.

17. (b)(i) $\mathbf{v} = (2y, -x)$; (v) $n = m = 2, A = \frac{1}{2}, f(x^2 + 2y^2) + \frac{1}{2}x^2y^2$;

(vi) $\frac{1}{2}(x^2 + 2y^2 - 2) + \frac{1}{2}x^2y^2$.

18. (b)(i) Fourier coeffs are sin: $b_n \cos\frac{2\pi nl}{L}$ and cos: $-b_n \sin\frac{2\pi nl}{L}$; (c)(i) $-\frac{K}{n\pi}(-1)^n$;

(ii) $\sum_{n=1}^{\infty} \frac{K(-1)^n}{n\pi} \sin\frac{2\pi nvt}{L} \cos\frac{2\pi nx}{L} - \frac{K(-1)^n}{n\pi} \cos\frac{2\pi nvt}{L} \sin\frac{2\pi nx}{L}$.

19. (a) a cube; (c)(i) $\frac{1}{2}$; (ii) $\cos a$; (iii) e^{2a}.

20. (b) $f = (A\cos\lambda x + B\sin\lambda x)(C\cos\lambda ct + D\sin\lambda ct)$, $f = (Ax + B)(Ct + D)$,

$f = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)\left(Ce^{\lambda ct} + De^{-\lambda ct}\right)$; (c) $f = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(C_n \cos\frac{n\pi ct}{L} + D_n \sin\frac{n\pi ct}{L}\right)$;

(d) $f = \sin\frac{\pi x}{L}\cos\frac{\pi ct}{L}$.

2017

Paper I

A1. (a) $-\ln 2$; (b) $y=2e^{-2}$. 2. (a) $-\frac{2}{3}\cos^3 x+c$; (b) $\frac{x^2}{2}\ln x-\frac{x^2}{4}+c$.

3. (a) $2^x \ln 2$; (b) $-\tan^2 t$. 4. (a) $-\frac{2}{3}$; (b) $y=-\frac{2}{3}x+\frac{5}{3}$.

5. (a) $x=2$; (b) $(x-2)(2x+1)(x+3)$. 6. (a) $(1+2x)^4$.

7. (a) $\sin^2\frac{\pi}{12}=\frac{1}{2}\left(1-\sqrt{1-\sin^2\frac{\pi}{6}}\right)$; (b) 0.067 . 8. (a) $x=\frac{\pi}{2}$ or $x=\frac{\pi}{6}$; (b) $x=\frac{3\pi}{2}$.

9. (a) $y=\frac{1}{c-x}$; (b) $y=\frac{1}{2-x}$. 10. (a) $OA=5$, $OB=13$; (b) 63/65 .

B11. (a) $z=re^{i\theta}$; (b) (i) $\Re=\frac{1}{2}\ln(x^2+y^2)$, $\Im=\tan^{-1}\frac{y}{x}+2n\pi$; (ii) $\Re=\frac{x}{x^2+y^2}$, $\Im=\frac{-y}{x^2+y^2}$;
(iii) $\Re=0$, $\Im=2y(x^2+y^2)$; (iv) $\Re=\cosh x\cos y$, $\Im=\sinh x\sin y$;
(c) (i) circles centred on origin; (ii) circles just touching the origin, centres on x-axis;
(d) $x=\cosh^{-1}2$, $y=(2n+1)\pi$, n integer.

12. (a)(ii) $\left(\frac{\partial x}{\partial y}\right)_z=\frac{xz}{\cosh(x+z)-yz}$, $\left(\frac{\partial x}{\partial z}\right)_y=\frac{xy-\cosh(x+z)}{\cosh(x+z)-yz}$, $\left(\frac{\partial y}{\partial z}\right)_x=\frac{\cosh(x+z)-xy}{xz}$;

(b) All a , b not both zero; (ii) $f=\tan^{-1}\left(\frac{by}{ax}\right)+c=-\tan^{-1}\left(\frac{ax}{by}\right)+k$; (iii) $y=Kx$.

13. (a) $y=ke^{-2x}+x^2-x+c$; (b) $y=e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}dx$; $y=e^{-3x^2/2}\left(3(x^2+1)^{3/2}-2\right)$.

14. (b)(i) Minimum at $(0,0)$, saddles at $\left(\frac{\pm 1}{\sqrt{(-a)}},0\right)$ if $a<0$; (iii) $y=\frac{\pm 1}{\sqrt{2}}(x^2-1)$.

15. (b) $9^{1/3}\approx 2+\frac{1}{12}-\frac{1}{(18)(16)}$, $\max R_n=\frac{5}{81(256)}$; (c)(i) $1+x+\frac{1}{2}x^2-\frac{1}{8}x^4$;

(ii) $1-\frac{x^2}{2}-x^3+\frac{5x^4}{24}$

16. (a) $N=(13)(48)$; (b) Binomial $\binom{100}{n}p^n(1-p)^{100-n}$, $\sum_{n=51}^{100}\binom{100}{n}p^n(1-p)^{100-n}$;

(c) (ii) $\frac{1}{2}\tanh\left(\frac{\mu}{2s}\right)$.

17. (a) (i) $\ln(\ln x)+c$; (ii) $\cosh x+sech\, x+c$; (b) $I_2=\frac{3\sqrt{\pi}}{4}$, $I_3=\frac{15\sqrt{\pi}}{8}$.

18. (a) $\boldsymbol{A}\boldsymbol{A}^T=\frac{1}{2}\begin{pmatrix}3&-1\\-1&3\end{pmatrix}$, $\boldsymbol{A}^T\boldsymbol{A}=\frac{1}{3}\begin{pmatrix}4&1&-2\\1&1&1\\-2&1&4\end{pmatrix}$; (b) $\lambda=1$ $\boldsymbol{u}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$,

$\lambda=2$ $\boldsymbol{u}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$; (c) $\sigma=0$ $\boldsymbol{v}=\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$, $\sigma=1$ $\boldsymbol{v}=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$,

$\sigma=2$ $\boldsymbol{v}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$; (d) $\boldsymbol{0}$, $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $\begin{pmatrix}1\\-1\end{pmatrix}$.

19. (a) ; (d) (i) diverges (ii) converges (using appropriate integral approximations is the least horrid method).

20. (a) $y^3=x^3+cx$; (b) $x^2+2y^2=k$.

Paper II

A1. (a) $90°$; (b) $\begin{pmatrix}1 & 0 & 2\end{pmatrix}^T$. 2. (a) $\Re = e^{-y}(x\cos x - y\sin x)$, $\Im = e^{-y}(x\sin x + y\cos x)$.

3. (a) $\lambda = \pm a$; (b) $\lambda = a$: $\boldsymbol{u} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$, $\lambda = -a$: $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$. 4. $x^3 - x^5$. 5. $\frac{3}{8} + \frac{1}{2}\cos 2x + \ldots$

6. $\begin{pmatrix}\sin(y-z) & \sin(z-x) & \sin(x-y)\end{pmatrix}^T$. 7. (a) $y = A\cos 3x + B\sin 3x$; (b) $y = \cos 4x$.

8. (a) $\mathbf{0}$; (b) $\frac{1}{12}$. 9. (a) $u''y^{-1} - \frac{1}{2}u'y^{-\frac{3}{2}}$. 10. (a) $\frac{(p+q)}{2}$; (b) $\frac{p}{(p+q)}$.

B11. (b) $|\boldsymbol{a} - \boldsymbol{b}| < p + q$ and $|\boldsymbol{a} - \boldsymbol{b}| > |q - p|$.

12. (a)(ii) $4\pi\rho_0 h_0\left(R^2 + 2Rh_0 + 2h_0^2\right)$; (b)(ii) $A = 1, B = \frac{1}{3}$.

13. (d) $\phi = x^2 y - y^2 - xz^3 - z + c$; (e) -126.

14. (a) $\mu = \sum_{x=0}^{\infty} xP(x)$, $\sigma^2 = \sum_{x=0}^{\infty} x^2 P(x) - \mu^2$; (b)(ii) $\sum_{x=0}^{K} \frac{e^{-\lambda}\lambda^x}{x!}$; (c) (i) $\left(1 - e^{\lambda}\right)^{10}$; (iii) $50\lambda^2 e^{-10\lambda}$.

15. (a)(i) $\mu = y^{-3}$, $x^2 - xy + ky^2 = 0$; (ii) $\mu = \cos x$, $y\cos^2 x + y^4 \cos^3 x = c$;
(b) $y = x \pm 2x^{1/2}$ if $x > 0$, or $y = Ax + 1/(A-1)$.

16. (b)(i) $-\frac{44}{3}$; (iii) $-\frac{44}{3}$.

17. (a) $\frac{1}{2}\begin{pmatrix}3 & -1 & 0\\-1 & 3 & 0\\0 & 0 & 8\end{pmatrix}$ $\lambda = 1$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}$, $\lambda = 2$ $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\lambda = 4$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$; (b) $\boldsymbol{y} = K\begin{pmatrix}1 & 1 & 0\end{pmatrix}^T e^{-t}$,
$\boldsymbol{y} = K\begin{pmatrix}1 & -1 & 0\end{pmatrix}^T e^{-2t}$, $\boldsymbol{y} = K\begin{pmatrix}0 & 0 & 1\end{pmatrix}^T e^{-4t}$; (c) $\boldsymbol{y_0} = K^{-1}\boldsymbol{f_0}$, $\boldsymbol{y_0} = \begin{pmatrix}1 & 1 & 1/4\end{pmatrix}^T$;
(d) $\boldsymbol{y} = \begin{pmatrix}1 & 1 & 0\end{pmatrix}^T e^{-t} + \frac{1}{4}\begin{pmatrix}0 & 0 & 1\end{pmatrix}^T e^{-4t}$.

18. (a) $\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2$; (b) $p = 3$.

19. (b) $V_p = \frac{R^2 H}{2}$; (c) the cylinder.

20. (a)(i) $y = \left(1 - \frac{1}{2}\ln(1+x)\right)^2$; (ii) $y = (1-3x)^{-\frac{1}{3}} e^{-\frac{x}{3}}$; (b) $T = Ce^{-\left(1 + \frac{n^2\pi^2}{L^2}\right)t}$; $X = B\sin\left(\frac{n\pi x}{L}\right)$ so general solution is $u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\left(1 + \frac{n^2\pi^2}{L^2}\right)t} \sin\left(\frac{n\pi x}{L}\right)$.

2018

Paper I

A1. (a) $x=y/\log_{10}2$; (b) 79 . 2. (a) $x=1,-1/2,-5$; (b) $-3/2-\sqrt{5}/2<x<-3/2+\sqrt{5}/2$.
3. $(t,\theta)=(0,\pi/2),(-1,0)$;. 4. (a) $1/9$; (b) $e^{\sin^2 x}+c$; 5. (a) $\theta=2\pi/3$; (b) $3\sqrt{3}/4\pi$ (Area B).
6. (a) $\sqrt{3}$; (b) $\cos\theta=1/3$. 7. (a) 1.094 ; (b) $9/2$. 8. $x=\pi/4\ y=e^{\pi/4}/\sqrt{2}$.
9. centre $(-1/2,-1)$ radius $3/2$. 10. (a) $(1,4)$; (b) $y=-2$.
B11. (a) $|z|=1/\sqrt{2}$, arg(z) = $5\pi/12$, $|w|=\sqrt{1/4\ln^2 2+\pi^2(5/12+2n)^2}$,
arg(w) = $-\tan^{-1}((5/12+2n)\pi/\frac{1}{2}\ln 2)+\pi$;
(b) $z=-i\pi/4+in\pi$; (c) $16\cos^5\theta-20\cos^3\theta+5\cos\theta$.
12. (a)(ii) $(\sqrt{2}-1)\sqrt{\pi}/2$; (iii) $(\sqrt{2}-1)\sqrt{\pi}$; (b)(i) $4\pi abc/3$ (ii) zero.
13. (a) $y=(\ln|x|+c)/x$; (b) $y=\frac{1}{4}e^{3x}+ce^{-x}$; (c) $5x^2/2+4xy-2y^4=c$; (d) $\ln|y|=2(x/y)^{1/2}+c$ or $y=0$.
14. (a) $y=1/4x$; (c) $(0,0),\pm(\sqrt{3}/2,\sqrt{3}/2),\pm(1/2,-1/2)$; (d) $(0,0)$ saddle, $\pm(\sqrt{3}/2,\sqrt{3}/2)$ minima, $\pm(1/2,-1/2)$ maxima.
15. (b) $1/a_0-a_1x/a_0^2+(a_1^2/a_0^3-a_2/a_0^2)x^2$; (c) $1+x^4/6$; (d) $\ln 2+1/2\,x+3/(8x^2)$
16. (a) $y_i=x_0\tan\theta_i$; (b) $g(\theta)=1/\pi$ for $-\pi/2\le\theta\le\pi/2$, zero otherwise, $G(\theta)=\theta/\pi+1/2$ for $-\pi/2\le\theta\le\pi/2$; (c) $F(y)=(1/\pi)\tan^{-1}(y/x_0)+1/2$; (f) $x_0\ln 3/\pi$; (g) $x_0\ln 3/\pi$.
17. (a) $(1/2)\ln|1-\cot^2(x)|+c$; (c) $E_{n+1}=(2n-1)E_n/2n$, $E_4=5\pi/32$.
18. (a) Det $\boldsymbol{A}$ = 0, Tr $\boldsymbol{A}$ = 6, Det $\boldsymbol{A}^T$ = 0, Det $\boldsymbol{A}^2$ = 0; (b) $\lambda=-6$ $\boldsymbol{u}=(1/\sqrt{2})(1\quad 0\quad -1)^T$, $\lambda=0$ $\boldsymbol{u}=(1/\sqrt{6})(1\quad 2\quad 1)^T$, $\lambda=12$ $\boldsymbol{u}=(1/\sqrt{3})(1\quad -1\quad 1)^T$; (c) $\boldsymbol{u}=k(1\quad 2\quad 1)^T\quad k\in\mathbb{R}$.
19. (a)(ii) m=0 not continuous or differentiable, m=1 continuous but not differentiable, m=2 continuous and differentiable; (b)(i) diverges; (ii) converges.
20. (a) $\tan^{-1}\alpha/\alpha$.

Paper II

A1. (a) $(1/\sqrt{5})(0 \quad 1 \quad 2)^T$; (b) $2/\sqrt{5}$. 2. $z = in\pi$, $z = i\pi(2n+1)$.

3. $2\cosh\phi$ or $2\sinh\phi$. 4. $x^2 - 3x^4/2$. 5. $\ln x - t$. 6. (a) $\boldsymbol{r} = \dfrac{\cos f(r)}{r}\dfrac{df}{dr}$; (b) $\dfrac{z}{r}\dfrac{df}{dr}$.

7. (a) $y = xe^{-x}$; (b) $y = 3/(x^3+3)$. 8. (a) 0 ; (b) π^2 . 9. (a) $\left(\pm\sqrt{\dfrac{3}{2}}, 0\right)$. 10. (a) $\alpha = 1/2$; (b) $\pi/2$.

B11. (b) $l = \sqrt{15}$; (c) $\boldsymbol{x} = (1/5)(2\boldsymbol{b} - 3\hat{\boldsymbol{n}} - \hat{\boldsymbol{n}}\times\boldsymbol{b})$.

12. (a) $\sin\phi$; (b) $\left(\dfrac{\partial\phi}{\partial y}\right)_x = \dfrac{\cos\phi}{r}$; (c) $f_y = g_r\sin\phi + g_\phi\dfrac{\cos\phi}{r}$;

(d) $\dfrac{\partial^2 f}{\partial y^2} = s^2 g_{rr} + \dfrac{2sc}{r}g_{r\phi} - \dfrac{2sc}{r^2}g_\phi + \dfrac{c^2}{r}g_r + \dfrac{c^2}{r^2}g_{\phi\phi}$, $\nabla^2 f = g_{rr} + \dfrac{g_r}{r} + \dfrac{g_{\phi\phi}}{r^2}$;

(e)(i) $f = 2xy$, $\nabla f = (2y, 2x)$; (iii) $3\sqrt{2}$.

13. (a)(i) $\int \boldsymbol{F}.d\boldsymbol{r} = 2$, $\int \boldsymbol{G}.d\boldsymbol{r} = 0$; (ii) $\int \boldsymbol{F}.d\boldsymbol{r} = 2\pi$, $\int \boldsymbol{G}.d\boldsymbol{r} = 0$, $\boldsymbol{G}$ is conservative,
$\Phi = x^2y^2 + c$; (b)(i) start $(1,0,0)$ end $(0,1,2\pi/3)$, helix; (ii) $5/6$.

14. (a)(i) $G\cap x_0, G\cap x_1, G\cap x_2, D\cap x_0, D\cap x_1, D\cap x_2$; (ii) pg_0 ; (iii) $pg_0 + (1-p)d_0$;
(iv) $d_1(1-p)/(pg_1 + (1-p)d_1)$; (v) $(g_1+g_2)p/((g_1+g_2)p + (d_1+d_2)(1-p))$;

(b)(i) $p\sum_{i=k+1}^{n} g_i + (1-p)\sum_{i=k+1}^{n} d_i$ if $k < n$; (ii) $p\sum_{i=0}^{k} g_i / [p\sum_{i=0}^{k} g_i + (1-p)\sum_{i=0}^{k} d_i]$;
(iii) $p = 1/(1+g_0)$, min p = 1/2 when $g_0 = 1$.

15. (a)(i) $y = e^x(A\cos x + B\sin x) + (x+1)^2$; (ii) $y = (A+x)e^{-x} + Be^{-2x}$; (b) $y = Ax\ln x + Bx$.

16. (a)(i) $6(\boldsymbol{a}.\boldsymbol{b})$; (ii) $\boldsymbol{0}$; (iii) $-2(\boldsymbol{a}.\boldsymbol{b})$; (iv) $\boldsymbol{a}\times\boldsymbol{b}$; (b)(i) $|\boldsymbol{a}\times\boldsymbol{b}|^2/2$; (ii) $2\pi R^3(\boldsymbol{a}.\boldsymbol{b})$.

17. (a) $\begin{pmatrix} 1 & \mu & 0 \\ 1 & -1 & 3 \\ 2 & -2 & \mu \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$; (b) c_i = i^{th} column of $\boldsymbol{A}$; (c) $\mu = -1, 6$ $\mu = -1$: $\lambda(1 \quad 1 \quad 0)^T$,

$\mu = 6$: $\lambda(-6 \quad 1 \quad 7/3)^T$; (d) $\lambda = 7$, (eg) $(1 \quad 0 \quad 2)^T + k(1 \quad 1 \quad 0)^T$.

18. (b) a_n and b_n are Fourier coeffs of f; (c) $a_n = 0$, $b_n = 4/(n\pi)$ if n is odd, $b_n = 0$ otherwise; (c)
$y_{f_0} = \sum_{n\,odd} r^4 \dfrac{4}{n\pi}\sin n\theta$; (d) $p = 2$.

19. (a)(i) $x = 2/(4-a^2), y = a/(4-a^2), z = (4-3a^2)/(4-a^2)^2$, no solutions if $a = \pm 2$;
(ii) Hessian is $\begin{pmatrix} -2 & a \\ a & -2 \end{pmatrix}$, so point is a max if $4 - a^2 > 0$ and a saddle if $4 - a^2 < 0$.

20. (a)(i) $u = \xi\eta/4 + A(\xi) + B(\eta)$; (ii) $a = b = d = 0, c = -1/2$; (b) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{2xz}{(z^2+x)^3}$, $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{-2z}{(z^2+x)^3}$.

2019

Paper I

A1. $-2 \le x \le 3/2$. 2. $(2,1)$. 3. (a) $c = \sqrt{3}/2 - \pi/6$; (b) -1 . 4. (a) $\sqrt{2}/2$; (b) $1/\sqrt{5}$.
5. $\pi/6$ or $5\pi/6$. 6. $I = y^3 \ln y/3 - y^3/9 + 1/9$. 7. (a) 10300 ; (b) 1364 . 8. $a = 2, b = -1$.
9. (a) 0 ; (b) $8|\boldsymbol{a}||\boldsymbol{b}| = \pi(|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2)$. 10. (a) 2 ; (b) $(\sqrt{3} - 5, 1 - 5\sqrt{3})$.

B11. (a) $8(\cos^6\theta - \sin^6\theta)$; (b) $\Re = \dfrac{\tan x(1 - \tanh^2 y)}{1 + \tan^2 x \tanh^2 y}$, $\Im = -\dfrac{\tanh y(1 + \tan^2 x)}{1 + \tan^2 x \tanh^2 y}$;
(c) $\left(\frac{x}{2}\right)^2 + \left(\frac{y-1}{2/3}\right)^2 = 1$ ellipse; (d) $r = e^{\theta}$.

12. (a) $R = \left(\dfrac{Ve}{2\pi(e-2)}\right)^{1/3}$; (b) $\dfrac{56}{15} bHL$ (c) $\pi b D^2 \left(1 - \dfrac{D}{4H}\right)$.

13. (a) $x^2 + xe^y - \sin y = e^{\pi/2}$; (b)(i) $y = ce^{-x^2/2} - 5e^{-x^2}$; (ii) $y = (x+1)x^{-x}$.

14. (a)(ii) $dm = -.04 m_A$, A has larger rest mass (b) $x^2 \dfrac{\partial^2 u}{\partial x^2} = x^2 y^2 \phi'' + \sqrt{xy}\left(\dfrac{-1}{4}\psi + \dfrac{y}{x}\psi' + \dfrac{y^2}{x^2}\psi''\right)$.

15. (b)(i) $1 - \dfrac{\pi^2 x^2}{8}$; (ii) $(x+1) - (x+1)^2 + \dfrac{7}{6}(x+1)^3 - \dfrac{7}{6}(x+1)^4$; (iii) $x^3 - \dfrac{x^5}{2} - \dfrac{x^6}{2}$.

16. (a) $4\alpha, 6\alpha, \beta, 3\beta, 6\alpha, 9\alpha, 3\beta, 3\beta$ where $\alpha = r/25, \beta = (1-r)/10$; (b)(i) $2/5$; (ii) $2/5$;
(iii) $2/5$; (c)(i) $(5+3r)/50$; (ii) $8r/(5+3r)$; (d) (N-1)r/(N-r).

17. (a)(i) $\dfrac{1}{8}e^{4x} + \dfrac{1}{4}e^{-2x} + c$; (ii) $\dfrac{-\arctan x}{x} + \ln|x| - \dfrac{1}{2}\ln(1+x^2) + c$; (b) ln 6;
(c) $I_n = -2n\pi^{2n-1} - 2n(2n-1)I_{n-1}$, $I_3 = -6\pi^5 + 120\pi^3 - 720\pi$.

18. (a)(i) $\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$; (ii) For $\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\boldsymbol{B} = \dfrac{1}{\sqrt{2}}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$, for $\boldsymbol{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\dfrac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$;
(b)(i) $\boldsymbol{M} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \end{pmatrix}^T$; (ii) $\alpha = \dfrac{\bar{y}\bar{x^2} - \bar{x}\bar{xy}}{\bar{x^2} - \bar{x}^2}$, $\beta = \dfrac{\bar{xy} - \bar{x}\,\bar{y}}{\bar{x^2} - \bar{x}^2}$.

19. (a)(i) diverges; (ii) converges to 4.

20. (b) $\int_{\sin x}^{\cos x} -t^4 e^{-xt^4} dt - \sin x\, e^{-x\cos^4 x} - \cos x\, e^{-x\sin^4 x}$; (c) $\dfrac{\pi}{4}\dfrac{1}{\sqrt{pq}}\left(\dfrac{1}{p} + \dfrac{1}{q}\right)$.

Paper II

A1. (a) $\dfrac{1/2(1+i)}{z+i} + \dfrac{1/2(1-i)}{z-i}$. 2. (a) $x^{3n}/n!$; (b) $x^{3n+1}/(3n+1)n!$. 3. $y = 3t/5 + ct^{-4}$.
4. (a) $I_{n+1} = x(\ln x)^{n+1} - (n+1)I_n$; (b) $x\ln x - x + c$. 5. (a) $(1,-1)$; (b) saddle point.
6. (a) $P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$; (b) 1.
7. $3/2$ or $1/2$. 8. $\dfrac{1}{4}\sin 2x + \dfrac{1}{8}\sin 4x$. 9. $\dfrac{\pi}{3}(4a^3 + 3a^4)$. 10. (a) $(0, 0, \pi a^2/2)$; (b) $\dfrac{1}{\sqrt{3}}\dfrac{\pi a^2}{2}$.

B11. (a)(ii) $1/4$; (iii) $\boldsymbol{f} + \boldsymbol{g} - \boldsymbol{e}$; (b) $\hat{\boldsymbol{n}} = (1,1,1)/\sqrt{3}$, $x + y + z = 1$, $1/\sqrt{3}$; (c) $\sqrt{3/8}$.
12. (a) $(0,0), (0,1), (\sqrt{3}/2, -1/2), (-\sqrt{3}/2, -1/2)$; (b) min, saddle, saddle, saddle respectively.
13. (a) $\alpha^2 + 2\alpha + \dfrac{5}{6}\beta$; (b) $\alpha^2 + 2\alpha + \dfrac{29}{35}\beta$; (c) $\beta = 0$; (d) $\phi = \alpha x^2 + \alpha^2 xy + \alpha xz$, $\alpha = -1 \pm \sqrt{2}$.

14. (a) $\int_a^b f(x)\,dx$; (b)(i) $\int_{a-c}^{b-c} f(x)\,dx$; (ii) $t(u)=f(u-c)$; (c)(i) $f(z-y)dz$;

(d)(i) $g(y)=1$ for $-1/2\le y\le 1/2$, zero otherwise; (ii) $h(z)=\frac{1}{\pi}\left(\arctan(z+1/2)-\arctan(z-1/2)\right)$; (iii) both zero.

15. (a)(ii) $A=\pi, B=e^{\pi}$; (ii) $y=\left(A+Bx+x^3/3\right)e^{-x}$; (b)(i) $\frac{d^2u}{dt^2}+2\frac{du}{dt}+2u=0$;

(ii) $u(t)=e^{-t}\left(A\cos t+B\sin t\right)$; (iii) $e^{-t}\left(\cos t-\sin t\right)$.

16. (a) $\pi\left(a^2b+ab^2+a^3\right)$; (b) e^{-1} .

17. (a) $\lambda=-1: \boldsymbol{x}=\begin{pmatrix}1 & -1 & 0\end{pmatrix}^T$, $\lambda=1: \boldsymbol{x}=\begin{pmatrix}0 & 0 & 1\end{pmatrix}^T$, $\lambda=3: \boldsymbol{x}=\begin{pmatrix}1 & 1 & 0\end{pmatrix}^T$;

(b)(i) $|\boldsymbol{A}||\boldsymbol{B}|$; (ii) $\alpha^n/|\boldsymbol{A}|$; (iii) $\left(\prod_i \beta_i\right)\left(\prod_j \gamma_j\right)|\boldsymbol{A}|$; (c)(i) $\boldsymbol{Q}=\boldsymbol{E}\,\boldsymbol{D}\,\boldsymbol{E}^{-1}$; (ii) $\boldsymbol{V}_k=\boldsymbol{V}_0\,\boldsymbol{E}\,\boldsymbol{D}^k\,\boldsymbol{E}^{-1}$.

18. (a) $a_n=\frac{2(-1)^n\sinh 1}{1+n^2\pi^2}$, $b_n=0$; $p=2, q=\pi$; (b) $2\sinh(1)\left(\frac{\sum_{n=1}^{\infty}(-1)^{n+1}n\pi\sin(n\pi x)}{1+n^2\pi^2}\right)$.

19. (a) $h=(4V/\pi)^{1/3}$, $r=(V/2\pi)^{1/3}$, $S=6\pi(V/2\pi)^{2/3}$; (b) $h=4R/3$, $r=2\sqrt{2}R/3$, $V_{cone}=\frac{8}{27}V_{sphere}$; (c) $x_i=a^{1/n}$, AM $\ge$ GM.

20. (a)(i) $Ae^{\lambda\left(x^2+2y\right)}$; (ii) $A\,x^{2\lambda}y^{\lambda}$; (b) $F'=-2HF\kappa$, $H'=-4H^2\kappa$, $T(x,t)=\frac{T_0L}{\sqrt{4\kappa t+L^2}}e^{-\left(\frac{x^2}{4\kappa t+L^2}\right)}$

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