2020

A1. x^2+x+1 . 2. $5\pi/4\le\theta\le 3\pi/2$. 3. $\sin(1/x)-(1/x)\cos(1/x)$. 4. $0<a<1$.

5. (a) $t^2+2t-1=0$; (b) $\sqrt{2}-1$. 6. $-(x/3)(3-2x)^{3/2}-(1/15)(3-2x)^{5/2}+c$. 7. $2\pi/3$.

8. -. 9. $ar(1-r^{100})/(1-r)$, $r\neq 1$. 10. 1.42.

B11. (a)(i) Re = 2/5, Im = 1/5; (ii) Re = cos(1), Im = sin(1); (iii) Re = 0, Im = $\pi/2+2n\pi$;

(iv) Re = 0, Im = 1; (b) $\left(\frac{y-1}{a/2}\right)^2+\left(\frac{x}{\sqrt{(a^2-4)}/2}\right)^2=1$ ellipse; (i) $\left(\frac{y-1}{2}\right)^2+\left(\frac{x}{\sqrt{3}}\right)^2=1$;

(ii) $x=0, 0\le y\le 2$.

12. (a)(i) $\sqrt{\pi}$; (ii) $3a^2\pi/2$ (b)(i) $\pi\sqrt{\pi}$ (ii) $\frac{2\pi}{3}(1-\cos\alpha)$.

13. (a)(i) $\sin x$; (ii) $y=\frac{-\cos^3 x}{3\sin x}+\frac{c}{\sin x}$; (b)(i) $y_c=e^{-\sqrt{3}x/2}(A\cos 3x/2+B\sin 3x/2)$;

(ii) $y_{p,1}=(1/3)e^{-\sqrt{3}x}$; (ii) $y_{p,2}=x/3-1/3\sqrt{3}$;

(iv) $y=e^{-\sqrt{3}x/2}\left((2/3\sqrt{3}-1/3)\cos(3x/2)+\left((2\pi/3)e^{\sqrt{3}\pi/2}\right)\sin(3x/2)\right)+e^{-\sqrt{3}x}/3+2x/3-2/3\sqrt{3}$.

14. (a) $f=1+(7/2)\ln 3$ at $(0,0)$, $f=(7/2)\ln 3$ at $(-1,0)$.

15. (b)(i) $2+1/80-1/(50(128))$; (c)(i) $1-x^2/3+x^4/18$;

(ii) $(x-1)-(1/2)(x-1)^2+(1/2)(x-1)^3-(1/2)(x-1)^4+\ldots$.

16. (a)(i) Both 1/2; (ii) 4/9; (iii) 4/9; (iv) 13/18; (v) 3.5; (b)(i) $\binom{10}{n}p^n(1-p)^{10-n}$; (ii) $\sqrt{10p(1-p)}$.

17. (a)(i) $\ln(2-\cos x)-\ln(3-\cos x)+c$; (ii) $-\ln x/(3x^3)-1/(9x^3)+c$;

(iii) $(x/2)\sqrt{1-x^2}+(\arcsin x)/2+c$; (b) $J=c^4/4$.

18. (a)(i) $(0,\pm 1,0),(0,\pm 1,2),(\pm 1,\pm 1,1)$; (ii) $(0,-1,0),(0,-1,2),(\pm 1,1,1)$; (b) $\lambda=\pm 3$;

(c)(i) $B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ or $B=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$, $C=\begin{pmatrix}-4 & -3\\ -3 & 4\end{pmatrix}$; (ii) $\lambda=5, \mathbf{v}=(1/\sqrt{10})\begin{pmatrix}1\\ -3\end{pmatrix}$,

$\lambda=-5, \mathbf{v}=(1/\sqrt{10})\begin{pmatrix}3\\ 1\end{pmatrix}$.

19. (a) 3; (b) $(b(\cos\theta)/a, b(\sin\theta)/a, 1)r\,dr\,d\theta$; (c) $\pi b a^2$; (d) $4\pi a^3$; (e) $\pi b a^2$.

20. (a) 1, 2/3; (c) $\phi=xy+yz+zx+c$; (e) three straight line segments, parallel to *x*, *y*, *z* axes respectively, works.

2021

Paper I

A1. $6x^2\sin x^3\cos x^3 - 4\sin x^3\sin 2x + 6x^2\cos x^3\cos 2x - 4\cos 2x\sin 2x$.

2. $(x-2)(x+1)$, $-1<x<2$. 3. $\theta=\pi/4, 3\pi/4$. 4. 4, 4/3. 5. Max $(1/2)e^{-1/2}$, Min 0.

6. $-\sqrt{2}/\sqrt{3}, 1/\sqrt{3}$. 7. Three. 8. $-1/4\, e^{-x^4}+c$ .9. $2t/(1-t^2),(1+t^2)^2/(1-t^2)^2$.

10. $y=-9x/16+25/16$.

B11. (a)(i) Re=-15/4, Im=2; (ii) Re=161/16, Im=-15;

(b)(i) hyperbolae z: (1/2,2) to (2,1/2), z*: (1/2,-2) to (2,-1/2); (ii) horizontal line (-15/4,2) to (0,2) to (15/4/,2); (iii) parabola (161/16,-15) to (0,-8) to (-4,0) to (0,8) to (161/16,15); (c) $\mathrm{Re}+4=(\mathrm{Im}/4)^2$.

12. (a)(i) $f(x)=1-x(1-e^{-1/x})$; (ii) $R/2r_0$ and $1-r_0/R$; (b) $\pi a^2 d/2$.

13. (a)(i) $y=A\cos x+B\sin x$; (ii) $y=1/2+1/6\cos 2x$;

(iii) $y=1/2+1/6\cos 2x-2/3\cos x-1/3\sin x$; (b) $y=0$ or $y=(ce^{-8x}-2)^{-1/4}$.

14. (a)(i) $dF=Gm_2\,dm_1/r^2+Gm_1\,dm_2/r^2-2Gm_1m_2\,dr/r^3$; (ii) 1%; (b) $-2\sqrt{2}c/9$; (c) r .

15. (c) $5/4+3/8(x-2)+(x-2)^2/16-(x-2)^3/32$; (d) $x<\ln 2$? (am not sure about this).

16. (a) Mean = 10.5, Var = 35/4; (b)(i) 1/36; (ii) 5/9; (iii) 5/12; (iv) 5/108; (c) 1/10.

17. (a) $f(x)$; (b)(i) $F=\ln(-x)$; (ii) $G=\ln|x|+c$; (c)(i) $F=\ln\left|x+\sqrt{x^2-1}\right|$;

(d) $F=\ln\left|\sin^2 x+\ln x\right|-\ln(\ln\pi)$.

18. (a)(i) $a=\pm\sqrt{3}$; (ii) any b except 0 or 3; (b)(i) $\boldsymbol{B}=\begin{pmatrix}1 & -2\\ -2 & 1\end{pmatrix}$; (ii) $\boldsymbol{B}^{-1}=-(1/3)\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$;

(d)(i) $b=0, c=0, d=\pm 1$; (ii) $b=0, c=0, d=1$; (e) $(1/2)\begin{pmatrix}2a & b+c\\ b+c & 6\end{pmatrix}+(1/2)\begin{pmatrix}0 & b-c\\ c-b & 0\end{pmatrix}$.

19. (a)(i) $p>-1$; (ii) $p\geq 0$; (b) $1/(3x^3)$ error $O(1/x^5)$.

20. (a) 1; (b) $\int_{e^{\alpha x}}^{e^{\beta x}} t^2\cos(xt^2)\,dt+\sin(xe^{2\beta x})\beta e^{\beta x}-\sin(xe^{2\alpha x})\alpha e^{\alpha x}$; equals zero when $\alpha=\beta$;

(c) $I=(1/2)\ln(\beta/\alpha)$.

Paper II

A1. (a) $\lambda=\cosh u+\sinh u:(1 \quad 1)^T, \lambda=\cosh u-\sinh u:(1 \quad -1)^T$. 2. $(-1 \quad 1 \quad 0)^T$.

3. $y=(1/3)e^{2x}+ce^{-x}$. 4. $\sqrt{(\pi/a)}$ and zero. 5. (a) $v=p/m, H=p^2/2m+V(x)$.

6. $(-xz \quad -yz \quad x^2+y^2)^T$, $-2z$. 7. $c=1-e^{-1}$, e^{-N}. 8. $(-1)^{n-1}x^{2(n-1)}$. 9. $a=-1$. 10. π.

B11. (a) $\sqrt{13}$; (b) $1/\sqrt{2}$; (c)(i) $\sqrt{(\boldsymbol{a}_2 . \boldsymbol{a}_2-(\boldsymbol{a}_2 . \hat{\boldsymbol{n}}_2)^2)}=|\boldsymbol{a}_2-(\boldsymbol{a}_2 . \hat{\boldsymbol{n}}_2)\hat{\boldsymbol{n}}_2|$; (ii) $\sqrt{2/3}$;

(d)(i) $d(t)=\sqrt{(t+1)^2+1/2}$; (ii) Asymptotes are $|t+1|$; (iii) $t_{min}=-1$, when $d(t)=1/\sqrt{2}$.

12. (b) $f(0,1)=1, f(0,-1)=-1$; (c) $f=0$ is $y=x^2$ a parabola, $f=1$ is $(0,1)$ a point,

$f=-1$ is $y=-1\pm x$; (0,1) is a maximum, (0,-1) is a saddle point.

13. (a) $(0 \quad 0 \quad 2a)^T$; (b) zero; (c) $a=0$; (d) $2a\pi$; (e) $2a\pi$; (f) $\phi=-e^{-r^2}/2$ when $a=0$.

14. (a) (ii) 1/2; (iv) 2; (b) $1/4-\pi/16$.

15. (a) $y=x-x/\ln x$; (b)(i) $d^2v/dt^2-4v=2$; (ii) $v=Ae^{2t}+Be^{-2t}-1/2$; (iii) $2Ae^{2t}-2Be^{-2t}$

(c)(i) $y=3x^2/4-3x/4+c+de^{-2x}$; (ii) $y=3x^2/4-3x/4+11/8-(3/8)e^{-2x}$.

16. (a)(i) $(ay/(x^2+y^2), -ax/(x^2+y^2), 0)$; (ii) zero; (iii) $\psi=a\ln r$, $\nabla\wedge\boldsymbol{A}=(0,-a/r,0)$; (b) 5/2.

17. (b)(i) $\boldsymbol{L}=\begin{pmatrix}1 & \lambda & 0\\ \lambda & 0 & -1\\ 0 & 4 & 1\end{pmatrix}$, $\boldsymbol{r}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$, $\boldsymbol{b}=\begin{pmatrix}1\\ 2\\ 1\end{pmatrix}$; (ii) solutions if $\lambda\neq\pm 2$;

(iii) $x=(3\lambda-4)/(\lambda^2-4), y=(\lambda-3)/(\lambda^2-4), z=(\lambda^2-4\lambda+8)/(\lambda^2-4)$; (c)(i) $\begin{pmatrix}a & -2c\\ c & a\end{pmatrix}$;

(ii) $\begin{pmatrix}0 & \mp 2\\ \pm 1 & 0\end{pmatrix}$.

18. (a) $a_n=\frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx$; (b) $a_0=L, a_n=0$ if n is even, $a_n=-4L/n^2\pi^2$ if n is odd;

(e) $\pi^4/96-1$.

19. (a)(i) $\begin{pmatrix}4 & 3\\ 3 & 2\end{pmatrix}$; (ii) Saddle point; (b) $\alpha=3^{1/4}m, \beta=3^{-1/4}m$.

20. (a) $\Phi=\frac{T_0}{2}-\frac{T_0}{2}e^{-4\pi^2\kappa t/L^2}\cos\frac{2\pi x}{L}$; (b) $\Phi=\frac{-8T_0}{\pi}\sum_{m=1}^{\infty}\frac{e^{-(2m-1)^2\pi^2\kappa t/L^2}}{(2m-1)[(2m-1)^2-4]}\sin\frac{(2m-1)\pi x}{L}$.

2022

Paper I

A1. (a) $(a-b)(a^2+b^2+ab)$; (b) $a+b$. 2. $x=-1$ or 3. 3. $x\le -1$ or $x>0$. 4. $x=0, y=0$.
5. (a) 50; (b) $1/(\sqrt{2}+1)$. 6. $x=3, y=27e^{-3}$. 7. -. 8. $e^{\sqrt{x}}+c$. 9. $(2,0), 1$. 10. $2(a^2+a)$; $\pi/3$.
B11. (a)(i) Re=0, Im= $-e^{-2m\pi}$; (ii) Re= $\cos\left((\pi/2+2n\pi)e^{-\pi/2-2m\pi}\right)$, Im= $\sin\left((\pi/2+2n\pi)e^{-\pi/2-2m\pi}\right)$;
(b) circle, centre $-1-i$, radius 8; (c)(i) ellipse, centre $-2-i$, semiaxes 16 and 8; (ii) ellipse, centre $2+2i$, semiaxes 16 and 8; (iii) ellipse, centre $-2+2i$, semiaxes 8 and 16;
(d) $16\sin^5\theta-20\sin^3\theta+5\sin\theta$, $\theta=0, \pi/6, 5\pi/6, \pi, 7\pi/6, 11\pi/6$.
12. (a)(i) Limits for x depend on y; (iii) and (iv) $4/9$; (b) $\pi\rho R^2 h(5.32/15)+\pi R^2\rho d$ (?)
13. (a) $y=\sqrt{-x^2-2x/3+cx^{-2}}$; (ii) $y=(1-\cos x)/\sin^2 x$.
14. (a)(i) $a=1$; (ii) $f=xy^2\sin x+c$; (b)(i) $f_x=2xf_u+2xy^2f_v$, $f_y=2yf_u+2yx^2f_v$; (ii)
$f_{xx}=4x^2f_{uu}+8x^2y^2f_{uv}+4x^2y^4f_{vv}+2f_u+2y^2f_v$, $f_{yy}=4y^2f_{uu}+8x^2y^2f_{uv}+4x^4y^2f_{vv}+2f_u+2x^2f_v$,
$f_{xy}=4xyf_{uu}+4xy(x^2+y^2)f_{uv}+4x^3y^3f_{vv}+4xyf_v$;
(iii) and (iv) $-\dfrac{4xy}{(1+x^2+y^2)^2}-\dfrac{4x^3y^3}{(1+x^2y^2)^2}+\dfrac{4xy}{(1+x^2y^2)}$.
15. (a) If $a=0$ then no Taylor series, if $a\ne 0$ then $f=|a|\left(1+\frac{1}{2}(x/a)^2-\frac{1}{8}(x/a)^4+\ldots\right)$;
(b)(i) $\ln(1+t(0))+\dfrac{xt'(0)}{(1+t(0))}$; (ii) $xt'(0)+x^2/2\left(t''(0)-[t'(0)]^2\right)$;
(iii) $xt'(0)+x^3/6\left(t'''(0)-[t'(0)]^3\right)$.
16. (a) $N=6-2e^{-2}$; (c) zero; (d) $\left(14/3-10e^{-2}\right)/\left(3-e^{-2}\right)$; (e) $M=2$.
17. (a)(ii) $-e^x\cos x+c$, $e^x\sin x+c$; (c)(i) $1/2$; (ii) $I(n)=\dfrac{n-1}{2}I(n-2)$;
(iii) $\sqrt{\pi}/4, 1/2, 3\sqrt{\pi}/8, 1, 15\sqrt{\pi}/16$.
18. (a) Tr $\boldsymbol{A}$ = 10, Det $\boldsymbol{A}$ = 0, Tr $\boldsymbol{A}^2$ = 52, Det $\boldsymbol{A}^2$ = 0; (b) $\lambda=0, \frac{1}{\sqrt{6}}(1,1,-2)$,
$\lambda=4, \frac{1}{\sqrt{2}}(1,-1,0)$, $\lambda=6, \frac{1}{\sqrt{3}}(1,1,1)$; (c) $\boldsymbol{x}=k(1,1,-2)+(2/3,-1/3,1/6)$.
19. (a)(i) diverges; (ii) converges to 9; (ii) does not converge; (iv) converges to e^2-1; (b) $\ln 2$.
20. (a) 1.

Paper II

A1. (a) $(-4 \quad 8 \quad -4)^T$; (b) $(0 \quad 0 \quad 1)^T$. 2. (a) $C=AB$; (b) $t=\mathrm{Tr}(A)$. 3. (a) No; (b) No; (c) Yes.

4. No. 5. $\lambda=\pm 2$. 6. $(1/2)\sin 2x+(1/2)\sin 4x$. 7. -. 8. $3x-6x^3$. 9. $(x^2/2)\ln x-x^2/4+c$.

10. $f_{xx}=2y^3, f_{xy}=f_{yx}=6xy^2, f_{yy}=6x^2y$.

B11. (a) $[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]\neq 0, \alpha=[\boldsymbol{y},\boldsymbol{v},\boldsymbol{w}]/[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}], \beta=[\boldsymbol{y},\boldsymbol{w},\boldsymbol{u}]/[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}], \gamma=[\boldsymbol{y},\boldsymbol{u},\boldsymbol{v}]/[\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}]$;

(b)(i) $\boldsymbol{x}=\boldsymbol{a}+\lambda\boldsymbol{c}$; (ii) if $\boldsymbol{b}.\boldsymbol{c}\neq 1: \lambda=\boldsymbol{a}.\boldsymbol{b}/(1-\boldsymbol{b}.\boldsymbol{c})$ unique value, if $\boldsymbol{b}.\boldsymbol{c}=1$ and $\boldsymbol{a}.\boldsymbol{b}=0$: any λ ,

if $\boldsymbol{b}.\boldsymbol{c}=1$ and $\boldsymbol{a}.\boldsymbol{b}\neq 0$: λ undefined ; (iii) if $\boldsymbol{b}.\boldsymbol{c}\neq 1: \boldsymbol{x}=\boldsymbol{a}+(\boldsymbol{a}.\boldsymbol{b}/(1-\boldsymbol{b}.\boldsymbol{c}))\boldsymbol{c}$ a point,

if $\boldsymbol{b}.\boldsymbol{c}=1$ and $\boldsymbol{a}.\boldsymbol{b}=0: \boldsymbol{x}=\boldsymbol{a}+\lambda\boldsymbol{c}$ a line.

12. (a) $\nabla f=(3x^2y+y^3-y, x^3+3xy^2-x)$; (b) $-3/8\sqrt{5}$; (c) saddles: $(0,0),(\pm 1,0),(0,\pm 1)$, minima: $\pm(1/2,1/2)$, maxima: $\pm(1/2,-1/2)$.

13. (a)(i) 5/4; (ii) 11/10; (iii) 5/4; (b) G: no, unless $\boldsymbol{a}=\boldsymbol{0}$, in which case $\phi=c$,

H: yes, $\phi=\int r f(r)\,dr+c$.

14. (a) (i) $10y^2(1-y)^3$; (ii) $1-(1-y)^5-5y(1-y)^4=1-(1-y)^4(1+4y)$;

(iii) $1-\left[(1-x)^5(1-y)^5+5x(1-x)^4(1-y)^5+5y(1-y)^4(1-x)^5\right]$

(b) $\dfrac{2x^2(1-y)^2+2y^2(1-x)^2}{2x^2(1-y)^2+2y^2(1-x)^2+5xy(1-x)(1-y)}$; (c) 3/100; (d) smallest integer above $\dfrac{\ln 0.05}{\ln(1-x)}$.

15. (a) $x\sin y=-\frac{1}{2}\cos 2y+c$, equivalent to $\sin y=(x\pm\sqrt{(x^2-A)})/2$; (i) $y=0$ or $y=\sin^{-1}x$

(ii) $y=\sin^{-1}x$; (iii) $y=\pi-\sin^{-1}x$ where in all three cases, $\sin^{-1}x$ is defined as an angle between $-\pi/2$ and $+\pi/2$; (b) $y(x)=(1/2)(1-e^{-3x})$, $y(1)=(1/2)(1-e^{-3})$.

16. (a) In case it is not clear: the cone is $(x,y,1-\sqrt{x^2+y^2})$ and the parabola is $(x,y,-1+x^2+y^2)$, and so all of x,y,z are within [-1, +1]; (b)(i) zero; (ii) $5\pi/6$.

17. (a) No; (b)(i) $\begin{pmatrix}1 & 2 & 3\\ 1 & 1 & a-1\\ a & 2 & 3\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$; (ii) $(1-a)(5-2a)$, vanishes for $a=1, a=5/2$;

(iii) $a=1, a=5/2$: no solutions, otherwise one solution.

18. (a)(i) $\sin nx, \cos nx$ for integer n; (ii) $\frac{1}{\pi}\int_{-\pi}^{\pi} e_n(x)e_m(x)\,dx=\delta_{nm}$; (iii) standard Fourier with $L=\pi$; (b) $f(x)=\sin x-2\sin 2x-\pi^2/6+\sum_{n=1}^{\infty}(1/n^2)\cos 2nx$.

19. (b) $\pm(1/\sqrt{2},0,-1/\sqrt{2})$ gives $T=0$, $\pm(1/2,\sqrt{2}/2,1/2)$ gives $T=\sqrt{2}/2$ max,

$\pm(1/2,-\sqrt{2}/2,1/2)$ gives $T=-\sqrt{2}/2$ min, ; (c) add $\mu(y-h(x))$ to Lagrangian function.

20. (a) $T=\sum(-2/n\pi)(-1)^n e^{-n^2\pi^2 x/a}\sin n\pi y$; (b) $a=0, b=1$ or $a=1/6, b=3/4$,

$h=-t$ or $h=(1/(6t^{1/3}))(1-x^2/t^{2/3})-3t/4$.

2023

Paper I

A1. 70, 56. 2. $(1\pm\sqrt{5})/2$. 3. $(0,1)$ and $(3,-2)$. 4. $(-1,-2)$ radius 3 . 5. $\phi=30°, A=2$.
6. $p=q=31$. 7. $x=-1, f=1/\sqrt{e}; x=1, f=\sqrt{e}$. 8. $dy/dx=-(1+e^x)/(1+e^y)$. 9. $n=2$.
10. $2-2\ln 2-(\ln 2)^2$.
B11. (a)(i) Re = 18, Im = 1; (ii) Re = 6/25, Im = -17/25; (b)(i) $y=x(\sqrt{2}-1)$ or $y=-x(\sqrt{2}+1)$;
(ii) $y=\pm x$; (c) $x=-\operatorname{arcosh}(2/\sqrt{3})=-\ln\sqrt{3}$; (d) $z=\ln(1+\sqrt{2})+i(\pi/2+2n\pi)$ or
$z=\ln(1-\sqrt{2})+i(\pi/2+2n\pi)$ or $z=\ln(\sqrt{2}-1)+i(-\pi/2+2n\pi)$.
12. (a)(i) $V=\pi R^2 H(3-e^{-2})/8$; (ii) $a=\sqrt[3]{3R^2H(3-e^{-2})/50}$; (b) $V=2DLH(2+\sqrt{2})/15$ (I originally got the same as the official answer, which is $V=4DLH(2+\sqrt{2})/15$, but now think that is wrong by the factor of 2).
13. (a) $y=(4/x+\ln|x|+c)^{-1}$; (b) $y=\tan x+\sqrt{2}\sec x$; (c)(i) $y=x((-1)^n\sin^{-1}(Ax)+n\pi)$;
(ii) $y=x\sin^{-1}(\sqrt{3}x/2)$; (iii) $y=x(\pi-\sin^{-1}(\sqrt{3}x/2))$
14. (a) $V=\sqrt{x}\left(1+\sqrt{1+y^2}\right)+c$; (b) $dV/V=2dr/r+dh/h$, $dV/V=0.04$.
15. (a) $x^3+x^7/3!+x^{11}/5!$; (b) $x-x^2+7x^3/6$; (c) $x^6-x^8+7x^{10}/15$.
16. (a) $A=12$; (c) $f(x)=12(x^2-x^3)\, for\, 0\le x\le 1$, zero otherwise; (e) Mode is 2/3;
(f) Median < Mode; (g) $E(X)=3/5, \sigma=1/5$;
(h) $12((4/5)^3/3-(4/5)^4/4-(2/5)^3/3-(2/5)^4/4)=0.64$.
17. (c) $a_n=\frac{4}{n^2\pi^2}(-1)^n, a_0=2/3, b_n=0$; (d) $b_n=\frac{2}{n\pi}(-1)^{n+1}, a_n=0$; (e) $A=4, B=90$.
18. (a) $\beta=1, \gamma=1-\alpha$; (b) $A_\alpha=\begin{pmatrix}1+\alpha & 1-\alpha & 2\\ 2 & \alpha & 2-\alpha\\ 2-\alpha & \alpha+3 & 1\end{pmatrix}$; (c) $10\alpha^2-10\alpha+8$.
19. (b)(i) converges; (ii) converges if $a<1$; (d)(i) zero; (ii) $a^a(\ln a+1)$;
(iii) $0\le a\le 1:1, 1<a\le 2:a, a>2:a^2/2$.
20. (b)(i) Minimise $\frac{1}{c}\left(\frac{2d_1}{\cos\theta}+\frac{\sqrt{2}d_2}{\cos\phi}\right)$ subject to $d_1\tan\theta+d_2\tan\phi=D$, $\sin\phi=\sqrt{2}\sin\theta$;
(ii) $\frac{\sqrt{3}\sin\theta}{\sqrt{(1-\sin^2\theta)}}+\frac{\sqrt{2}\sin\theta}{\sqrt{(1-2\sin^2\theta)}}=2$, $\sin\theta=1/2$; (iii) $\theta=30°\ \phi=45°$, time taken = $6/c$,
distance = $2+\sqrt{2}$.

Paper II

A1. (a) $X=A$; (b) $Y=\Omega$. 2. (a) $r=5,\theta=\arctan(4/3)+\pi,z=-1$;

(b) $r=\sqrt{26},\theta=\pi-\arctan(5),\phi=\arctan(4/3)+\pi$. 3. (a) $\alpha=0,\beta=\boldsymbol{a}.\boldsymbol{c},\gamma=-(\boldsymbol{a}.\boldsymbol{b})$;

(b) Area of parallelogram formed by $\boldsymbol{a}$ and $\boldsymbol{b}$. 4. $(1,-1,0)/2$. 5. Cumulative normal distribution.

6. (a) $x_1=x_0-f(x_0)/f'(x_0)$; (b) 3/2. 7. (a) $(-1,-1,-1)$; (b) $-\sqrt{2}$. 8. (a) zero; (b) zero.

9. (a) $(\boldsymbol{a}.\boldsymbol{c})(\boldsymbol{a}.\boldsymbol{b})$; (b) $3(\boldsymbol{b}.\boldsymbol{b}+1)$. 10. (a) $z=-x-y$ any x,y ; (b) Plane $x+y+z=0$.

B11. (c)(i) No; (ii) Yes.

12. (a) $x=\ln(c/a),y=\ln(d/b),h=c+d-c\ln(c/a)-d\ln(d/b)$;

(b) $c>0,d>0:\text{min}, c \text{ and } d \text{ different signs}: \text{saddle}, c<0,d<0:\text{max}$; (c)(i) Min at $(0,0)$;

(ii) Saddle at $(0,0)$.

13. (a)(i) $\boldsymbol{F}:a,\boldsymbol{G}:\sin(1)+c$; (ii) $\boldsymbol{F}:a,\boldsymbol{G}:\sin(1)+c$; (b) $b=0,\text{any } a,c$.

14. (a)(i) $6!$; (ii) $5!$; (iii) $2(5!)=240$; (iv) $4!$; (v) $2(5!)-4!=216$; (vi) $2(5!)-2(4!)$; (b)(i) one;

(ii) $(4!)(2!)$; (iii) ${}^6C_4$; (v) For {m,6-m}, (ii) is $m!(6-m)!$, (iii) is ${}^6C_m$; (iv) is $6!$.

15. (a) if $k=1$ $y=A\cos x+B\sin x+(x/2)\sin x$, if $k\neq 1$ $y=A\cos x+B\sin x+(\cos kx)/(1-k^2)$;

(b)(i) $\alpha=a+d,\beta=ad-bc$; (ii) $u=-te^{2t},v=(1-t)e^{2t}$.

16. (a) $3+e^{-1}$; (b)(i) $dS=\sqrt{2}\,dx\,dz$; (ii) $\sqrt{2}\pi(2\pi+5/3)$; (c)(i) 4π ; (ii) zero.

17. (b) $Q=\begin{pmatrix}1/\sqrt{3} & 0 & 2/\sqrt{6}\\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6}\\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\end{pmatrix}$; (c)(i) $\frac{1}{3}\begin{pmatrix}1 & -1\\ 1 & 2\end{pmatrix}$; (ii) $\begin{pmatrix}1 & 0\\ 0 & 4\end{pmatrix}$; (iv) $\begin{pmatrix}1366 & 2730\\ 1365 & 2731\end{pmatrix}$.

18. (a)(i) $\int\sin^6 x\,dx=\frac{5x}{16}-\cos x\left(\frac{1}{6}\sin^5 x+\frac{5}{24}\sin^3 x+\frac{5}{16}\sin x\right)+c_1$,

$\int\cos^6 x\,dx=\frac{5x}{16}+\sin x\left(\frac{1}{6}\cos^5 x+\frac{5}{24}\cos^3 x+\frac{5}{16}\cos x\right)+c_2$; (b) $\pi/4, 3\pi/16, \pi/4, 3\pi/16$.

19. (a) Equality when $f=-\lambda g$; (b)(i) $K=\pi h\int_{r=0}^{a} r^3\omega^2(r)\,dr$; (ii) $L=2\pi h\int_{r=0}^{a} r^3\omega(r)\,dr$,

$f=\sqrt{(\pi h r^3)}\,\omega(r),g=2\sqrt{(\pi h r^3)}$; (iii) $\omega(r)=constant$.

20. (a) $u=Ae^{kx}(2y-6)^{k/2}$; (b) $u=Ay^k\exp(-k(x+1)e^{-x})$; (c) $u=e^{-2\cos x}\sin^2 y$.

2024

Paper I

A1. (a) $\sqrt{6}$; (b) 150°. 2. $y=3x-2$. 3. $-2xe^{2-x^2}$, $f=e^2$. 4. (a) 780; (b) 6.
5. $3/2-\sqrt{5}/2\le x\le 3/2+\sqrt{5}/2$. 6. (a) 2; (b) $(1,0)$ and $(0,1)$. 7. $x=30°$.
8. $\text{radius}=2,\text{centre}=(1,-1/2)$. 9. $n=-2,a=1$. 10. (a) $x\sin x+\cos x+c$; (b) 2.
B11. (d)(i) dot product; (ii) area of parallelogram; (f) N.
12. (b) $2\pi\gamma\rho a^3(\beta^2-(\alpha-\gamma)^2)$; (c) $\frac{4}{3}\pi\rho\beta^3a^3-\pi\rho a^3\left(\frac{2}{3}\beta^3-\beta^2(\alpha-\gamma)+(\alpha-\gamma)^3/3\right)$, ie

$\pi\rho a^3\left(\frac{2}{3}\beta^3+\beta^2(\alpha-\gamma)-(\alpha-\gamma)^3/3\right)$; (d) as (c); (e) it's (b)+2(c) obviously.

13. (a) $y=(4/x+x^3/3+c)^{-1}$; (i) $y=(4/x+x^3/3-10/3)^{-1}$; (ii) $y=0$; (b)(i) $\mu=\frac{1}{\sqrt{16+x^2}}$;

(ii) $y=\frac{16+x^2}{3}+\frac{c}{\sqrt{16+x^2}}$.

14. (a) $dc/c=\frac{a\,da+b\,db}{a^2+b^2}$, $dA/A=da/a+db/b$, -0.01, so 1% decrease.

15. (a) $\ln(1.1)\approx 0.095$, upper bound 1/3000 ; (c)(i) $1-x^2/4+7x^4/96$;
(ii) $e-(e/2)(x-\pi/2)^2+(e/6)(x-\pi/2)^4$.

16. (a)(ii) $1-2p_w$; (b)(i) $2x-2p_w$; (ii) $\frac{1-2x+p_w}{1-2x+2p_w}$; (c)(i) $N=20$; (ii) $\binom{10}{4}\binom{10}{8}/\binom{20}{12}$.

17. (a) $2+2\ln 2$; (b) $1/2-1/(1+\sqrt{3})$; (c) $(1/2)\ln(e^{2x}+1)-2\arctan(e^x)+c$;

(d) $(1/2)\arctan\left(\frac{\tan x}{2}\right)+c$.

18. (a)(i) $\lambda=-2,\boldsymbol{v}=(0\quad 1\quad -1)^T/\sqrt{2}$, $\lambda=4,\boldsymbol{v}=(1\quad 0\quad 0)^T$, $\lambda=6,\boldsymbol{v}=(0\quad 1\quad 1)^T/\sqrt{2}$;

(ii) $(1/2)\begin{pmatrix}2 & -3 & 1\\ 0 & -1 & 1\\ -2 & 7 & -3\end{pmatrix}$ $\boldsymbol{x}=(1/2)(-1\quad 1\quad 3)^T$; (b)(i) reflection in one of the xy-plane, the xz-plane, or the yz-plane.

19. (a) $e^x(x^2+200x+9900)$; (b)(ii) $p>0$; (iii) $p>1$, $f'(0)=0$ (c)(i) converges; (ii) diverges.

20. (b)(i) and (ii) $6\pi^2x^5-4x^3+\cos x-\cos\pi x^2+2\pi x^2\sin\pi x^2-x\sin x$; (c)(i) $\frac{dg}{dx}=\frac{g}{x+1}$;

(iii) $g=x+1$; (iv) 1.

Paper II

A1. (a) 1; (b) all zero. 2. (a) $r=\sqrt{2}$; (b) $\theta=\pi-\arctan(\sqrt{2})$. 3. $\xi=1/\ln 2$. 4. (a) $f(x)$; (b) $\cos x(1-\cos x)+\sin^2 x$. 5. $\alpha=-1, \beta=-1$. 6. (a) zero; (b) $P(A)+P(B)$. 7. $f=e^{xy}-1$. 8. (a) 1; (b) zero. 9. $a_{21}=4, a_{22}=4$. 10. $(1/2)\sin 2x+\sin x$.

B11. (a) $R_{\hat{u}}(\boldsymbol{r})=\boldsymbol{r}-P_{\hat{u}}(\boldsymbol{r})$, $P_{\hat{u}}(\boldsymbol{r})$ and $R_{\hat{u}}(\boldsymbol{r})$ are two sides of a right-angled triangle which has $\boldsymbol{r}$ as its hypotenuse, with $P_{\hat{u}}(\boldsymbol{r})$ parallel to $\hat{u}$; (b) $\boldsymbol{r}$; (c) eg $\boldsymbol{p}=\boldsymbol{y}, \hat{\boldsymbol{q}}=\frac{\boldsymbol{z}-\boldsymbol{y}}{|\boldsymbol{z}-\boldsymbol{y}|}$ or equivalent; (d)(ii) $\hat{\boldsymbol{a}}$ is parallel to the line; (iii) c = distance of line from origin; (iv) the line and the origin; (e) $\boldsymbol{r}=\hat{\boldsymbol{g}}\times\boldsymbol{f}+t\hat{\boldsymbol{g}}$.

12. (a) $(6x^5-48xy^2, 6y^5-48x^2y)$; (b) $y=0$ or $x=\frac{\pm 1}{\sqrt{8}}y^2$, $x=0$ or $y=\frac{\pm 1}{\sqrt{8}}x^2$; (c) $(0,0),(\pm\sqrt{8},\pm\sqrt{8})$; (d) minima; (e) saddle.

13. (a) no for $\boldsymbol{G}$, yes for $\boldsymbol{K}$, $\phi=(1/2)\ln(1+r^2)+c$; (b)(i) $(4+\mu+\lambda)/3$; (ii) $\lambda+5/3$; $\mu=2, \lambda=1/2$.

14. (a)(ii) mean = 1/2, variance = 1/12; (b)(iii) mean = 0, stdev = $1/\sqrt{6}$; (iv) 1/8.

15. (a) $\gamma=4$; (ii) $y=Ae^{-x}+Be^{-5x}$; (iii) $y=e^{-3x/2}\left(A\cos\frac{\sqrt{7}x}{2}+\sin\frac{\sqrt{7}x}{2}\right)$; (b) $y=25x^2+40x+22-21e^{2x}\cos x$.

16. (a) $(2yx+4x^2y^2z^2+x^4z^2+x^4y^2)e^{x^2yz}$; (b)(ii) $OAC:-1/4, OBC:-1/3, ABC:1$; (iii) 1/4.

17. (b) $3(2\alpha+1)(\alpha-3)$, zero when $\alpha=3$ or $\alpha=-1/2$, any β ; (c)(i) $x=(1-2\alpha)/(1+2\alpha), y=2/(1+2\alpha), z=0$; (ii) $x=-5/7-(18+\beta)z/7, y=2/7-(3-\beta)z/7$, any z ; (iii) no solution.

18. (c)(i) $f(x)=\sum_{odd\ n}\frac{8}{n^3\pi^3}\sin n\pi x$; (ii) $A=1/960$.

19. (a) $r=\sqrt{2/3}R, h=2R/\sqrt{3}$, fraction is $1/\sqrt{3}$; (b)(i) $\max f=1 \text{ at} \pm\left(\frac{1}{\sqrt{8}}, \sqrt{8}\right)$, $\min f=-1 \text{ at} \pm\left(\frac{1}{\sqrt{8}}, -\sqrt{8}\right)$.

20. (b) $\psi=1/2+(1/2)\cos x\cos vt$; (c)(i) same as (b).